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ELEMENTARY GEOMETRY

AND

CONIC SECTIONS.

(*Edition for Indian Schools*)

ELEMENTARY GEOMETRY

AND

CONIC SECTIONS

BY

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ELEMENTARY GEOMETRY.

BOOKS I—V

"'Ὡς οἷόν τ' ἄρα, ἦν δ' ἐγώ, μάλιστα προστακτέον, ὅπως οἱ ἐν τῇ καλλιπόλει σοι μηδενὶ τρόπῳ γεωμετρίας ἀφέξονται πρὸς γὰρ πάσας μαθήσεις, ὥστε κάλλιον ἀποδέχεσθαι, ἴσμεν που ὅτι τῷ ὅλῳ καὶ παντὶ διοίσει ημμένος τε γεωμετρίας καὶ μή   τῷ παντὶ μέντοι νὴ Δί', ἔφη"

PLATO, *Repub* Bk VII 527

This was Diuine Plato his Iudgement, both of the purposed, chief, and perfect vse of Geometrie; and of his second, dependyng and deriuatiue commodities. And for vs, Christen men, a thousand thousand mo occasions are to haue nede of the helpe of Megethologicall Contemplations; wherby to trayne our Imaginations and Myndes, by litle and litle, to forsake and abandon the grosse and corruptible Objectes of our vtward senses: and to apprehend, by sure Doctrine demonstratiue, Things Mathematicall.

John Dee his Mathematicall Preface to Euclides Elementes
A. D. 1570.

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INTRODUCTION

THE Science of *Geometry* treats of the properties and construction of *solids*, *surfaces*, and *lines*. *Plane* Geometry treats only of the *line* and *plane* or flat surface, and the *elements* of Plane Geometry include the properties of the *straight* line and *circle* only, and of combinations of straight lines and circles

The science of Geometry is called *deductive*, because certain fundamental truths being assumed as obviously true, the remaining truths of the science are deduced from them by reasoning

Propositions admitted without demonstration are called *Axioms*.

Of the Axioms used in Geometry those are termed *General* which are applicable to magnitudes of all kinds · the following is a list of the general axioms more frequently used

(*a*) The whole is greater than its part

(*b*) The whole is equal to the sum of its parts

(*c*) Things that are equal to the same thing are equal to one another.

(*d*) If equals are added to equals the sums are equal

(*e*) If equals are taken from equals the remainders are equal.

(*f*) If equals are added to unequals the sums are unequal, the greater sum being that which is obtained from the greater magnitude

(*g*) If equals are taken from unequals the remainders are unequal, the greater remainder being that which is obtained from the greater magnitude.

(*h*) The doubles and halves of equals are equal

A *Theorem* is the formal statement of a proposition that may be demonstrated from known propositions. These known propositions may themselves be Theorems or Axioms.

The two next pages, within brackets, may be omitted the first time of reading the subject

[A Theorem consists of two parts, the *hypothesis*, or that which is assumed, and the *conclusion*, or that which is asserted to follow therefrom. Thus in the typical Theorem

If A is B, then C is D, (i)

the hypothesis is that A is B, and the conclusion, that C is D.

From the truth conveyed in this Theorem it necessarily follows.

If C is not D, then A is not B, (ii).

Two such Theorems as (i) and (ii) are said to be *contrapositive*, each of the other.

For example, if it were universally true that, If a man is a Spaniard, his hair is black, then it would follow that if his hair is not black, the man is not a Spaniard Each of these statements is the contrapositive of the other.

Two Theorems are said to be *converse*, each of the other, when the hypothesis of each is the conclusion of the other.

Thus,

If C is D, then A is B, (iii)

is the converse of the typical Theorem (i)

The contrapositive of the last Theorem, viz

If A is not B, then C is not D, (iv)

is termed the *obverse* of the typical Theorem (i)

Sometimes the hypothesis of a Theorem is complex, i e consists of several distinct hypotheses, in this case every Theorem formed by interchanging the conclusion and *one* of the hypotheses is a converse of the original Theorem

The truth of a converse is not a logical consequence of the truth of the original Theorem, but requires independent investigation

Thus, supposing it were true that if a man is a Spaniard his hair is black; it does not follow that if a man's hair is black he is therefore a Spaniard for he might be a Turk or of any other nation

Hence the four associated Theorems (i) (ii) (iii) (iv) resolve themselves into two Theorems that are independent of one another, and two others that are always and necessarily true if the former are true, consequently it will never be necessary to demonstrate *geometrically* more than two of the four Theorems, care being taken that the two selected are not contrapositive each of the other

Rule of Conversion If of the hypotheses of a group of demonstrated Theorems it can be said that one must be true, and of the conclusions that no two can be true at the same time, then the converse of every Theorem of the group will necessarily be true

Obs. The simplest example of such a group is presented when a Theorem and its obverse have been demonstrated, and the validity of the rule in this instance is obvious from the circumstance that the converse of each of two such Theorems is the contrapositive of the other. Another example, of frequent occurrence in the elements of Geometry, is of the following type·

If A is greater than B, C is greater than D.
If A is equal to B, C is equal to D
If A is less than B, C is less than D.

Three such Theorems having been demonstrated *geometrically*, the converse of each is always and necessarily true.

Rule of Identity. If there is but one A, and but one B, then from the fact that A is B it necessarily follows that B is A.

This is an important axiom in geometrical reasoning De Morgan used to illustrate it by the following example —

Suppose that in a town there were *only one* post-office and *only one* grocer's. and that it was known that the post-office was the grocer's; then it would follow that the grocer's was the post-office.

This is called the axiom of the unique solution, or the rule of identity]

Explanation of Terms and Signs.

A *Problem* is a geometrical construction to be effected by the aid of certain instruments.

It has been universally agreed by Geometers to use only *the ruler*, i e. a straight edge, not divided, and a *pair of compasses*, in the solution of Problems.

A *Corollary* is a geometrical truth easily deducible from a theorem.

Q. E D stands for *quod erat demonstrandum*, and is usually written at the end of a theorem to mark that the truth of the theorem has been proved

The parts of a Theorem are the *general enunciation* of the hypothesis and the fact to be proved, or statement in general language, the *particular enunciation*, or statement of the hypothesis and the fact to be proved in the particular case examined, and the *proof*.

In the proof it is frequently necessary to draw certain lines, or to conceive them as drawn This is called the *construction*

REMARKS

A beginner often asks 'What is the use of Geometry?'

The following remarks may perhaps help to shew him part at least of the use of it.

What *is* Geometry? What is the object of the science?

It is not *measurement*, because that may be done *directly* If I want to find the height of a tower, I may go to the top, and let a string down to the bottom, and then measure the string, but this is not geometry, though it is measurement. Geometry is the science of *indirect measurement*, in which, for example, by measuring one line we learn the length of another If I measure the length of the *shadow* of the tower, and also the length of a vertical stick and its shadow, and have proved by geometrical reasoning, that as the length of shadow of the stick is to the length of shadow of the tower, so is the height of the stick to the height of the tower, that is, *measure the height of the tower indirectly*, this is a geometrical operation

Now it is plain that many measurements *must* be effected indirectly How for example is the height of a mountain ascertained? Or how is the distance of the moon from the earth found out to be very nearly 238000 miles? How do we know approximately the size of the sun, or the weight of some of the stars, or the velocity of light? It is plain that these results must be obtained by indirect measurement; and some of

them are obtained by measurement extremely indirect and circuitous, and consisting of a very great number of successive steps of reasoning, each result, as soon as it is obtained, serving as the starting-point from which fresh results are attainable.

Now Elementary Geometry gives the beginning of all such chains of reasoning The theorems are results which follow from the axioms, and, in their turn, will serve as the foundations for fresh theorems arranged in a long chain until questions such as those above mentioned can be solved.

Every theorem therefore may be shewn to be a means of indirectly measuring some magnitude. In theorem 4, for example, it is proved that $AOD = COB$; that is, if AOD is accessible, and is measured (by an instrument suitable for measuring angles), then it is not necessary to measure COB, for you have proved that it will be the same as AOD.

Again in Theorem 7, let A be a post on one bank of a river, B, C two posts on the opposite bank; it is required to find the distance across the river from B to A.

Measure BC, (which you can do, as they are both on the same bank,) and put up two posts E, F in a field at the same distance apart that B is from C: measure the angle at B, that is how much, when standing at B, you must turn a line pointing at C till it points at A; and copy this angle at E similarly measure the angle at C, and copy it at F. Then this theorem has proved that $AB = DE$, that is if you measure in the field ED, you will indirectly have measured AB

Theorem 5 is of very great importance, and is a good illustration of indirect measurement Suppose B and C are two points with an obstacle between them, a house or a hill for example, how is the distance from B to C to be measured? This theorem tells you; you may think it out for yourself

So with this clue to the practical application of the theorems it will be well to go through all of them; finding out in each case what the magnitude is which is indirectly measured, or the result indirectly obtained; and inventing practical questions to which each theorem could be applied.

BOOK I.

THE STRAIGHT LINE.

DEFINITIONS

√*Def.* 1. A *point* has position, but it has no magnitude.

√*Def.* 2. A *line* has position, and it has length, but has neither breadth nor thickness. The extremities of a line are points, and the intersection of two lines is a point

√*Def* 3. A *surface* has position, and it has length and breadth, but not thickness The boundaries of a surface, and the intersection of two surfaces, are lines

Def 4. A *solid* has position, and it has length, breadth and thickness.

The boundaries of a solid are surfaces.

Def 5 A *straight line* is such that any part will, however placed, lie wholly on any other part, if its extremities are made to fall on that other part

√*Def.* 6. A *plane surface*, or *plane*, is a surface in which any two points being taken the straight line that joins them lies wholly in that surface.

√*Def* 7. A plane *figure* is a portion of a plane surface enclosed by a line or lines.

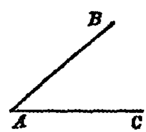
√*Def.* 8. A *circle* is a plane figure contained by one line, which is called the *circumference*, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another. This point is called the *centre* of the circle.

√*Def* 9. A *radius* of a circle is a straight line drawn from the centre to the circumference.

√*Def.* 10. A *diameter* of a circle is a straight line drawn through the centre and terminated both ways by the circumference.

√*Def* 11. When two straight lines are drawn from the same point, they are said to contain, or to make with each other, a *plane angle*. The point is called the *vertex*, and the straight lines are called the *arms*, of the angle.

An *angle* is a simple concept incapable of *definition*, properly so called, but the nature of the concept may be explained as follows, and for convenience of reference it may be reckoned among the definitions



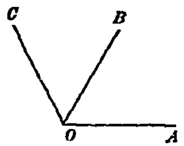
A line drawn from the vertex and turning about the vertex in the plane of the angle from the position of coincidence with one arm to that of coincidence with the other is said to turn through the angle: and the angle is greater as the quantity of turning is greater Since the line may turn from the one position to the other in either of two ways, two angles are formed by two straight lines drawn from a point. These angles (which have a common vertex and common arms) are said to be *conjugate*. The greater of the two is called the *major conjugate*, and the smaller the *minor conjugate*, angle.

When *the angle contained by two lines* is spoken of without qualification, the *minor conjugate* angle is to be understood. It is seldom requisite to consider major conjugate angles before Book III.

When the arms of an angle are in the same straight line, the conjugate angles are equal, and each is then said to be a *straight angle*

An angle is named by a single letter at its vertex, as A or by a letter at the vertex placed between letters at points on each of its arms, as BAC, or CAB

Def 12 When three straight lines are drawn from a point, if one of them be regarded as lying between the other two, the angles which this one (the mean) makes with the other two (the extremes) are said to be *adjacent* angles and the angle between the extremes, through which a line would turn in passing from one extreme through the mean to the other extreme, is the sum of the two adjacent angles.



Thus AOB, BOC are adjacent,

and $AOB + BOC = AOC$,

also $AOC - COB = AOB$

Def. 13 The *bisector* of an angle is the straight line that divides it into two equal angles.

Def 14 When one straight line stands upon another straight line and makes the adjacent angles equal, each of the angles is called a *right angle*

Obs Hence a straight angle is equal to two right angles, or, a right angle is half a straight angle, and a straight line makes with its continuation at any point an angle of two right angles.

Def 15. A *perpendicular* to a straight line is a straight line that makes a right angle with it.

Def. 16 An *acute* angle is that which is less than a right angle.

√*Def.* 17. An *obtuse* angle is that which is greater than one right angle, but less than two right angles.

Def. 18. A *reflex* angle is a term sometimes used for a major conjugate angle

√*Def.* 19. When the sum of two angles is a right angle, each is called the *complement* of the other, or is said to be *complementary* to the other.

√*Def.* 20. When the sum of two angles is two right angles, each is called the *supplement* of the other, or is said to be *supplementary* to the other.

√*Def.* 21. The opposite angles made by two straight lines that intersect are called *vertically opposite angles.*

√*Def* 22. A *plane rectilineal figure* is a portion of a plane surface inclosed by straight lines. When there are more than three inclosing straight lines the figure is called a *polygon.*

√*Def.* 23. A polygon is said to be *convex* when no one of its angles is reflex.

Def. 24. A polygon is said to be *regular* when it is equilateral and equiangular, that is, when all its sides and angles are equal

√*Def.* 25. A *diagonal* is the straight line joining the vertices of any angles of a polygon which have not a common arm.

Def. 26 The *perimeter* of a rectilineal figure is the sum of its sides.

Def. 27. The *area* of a figure is the space inclosed by its boundary.

√*Def.* 28. A *triangle* is a figure contained by three straight lines.

Def 29 A *quadrilateral* is a polygon of four sides, a *pentagon* one of five sides, a *hexagon* one of six sides, and so on.

GEOMETRICAL AXIOMS

1 Magnitudes that can be made to coincide are equal.

2 Two straight lines that have two points in common lie wholly in the same straight line

3. A finite straight line has one and only one point of bisection

4 An angle has one and only one bisector

POSTULATES

Let it be granted that

1. A straight line may be drawn from any one point to any other point

2 A terminated straight line may be produced to any length in a straight line

3 A circle may be described from any centre, with a radius equal to any finite straight line

It will be seen that these postulates amount to a request to use the straight edge of a ruler, and a pair of compasses, the latter being such that a distance can be carried by them from one part of the paper to another

It may be useful to have a list of the derivations of some of the common terms used in geometry

Axiom ἀξίωμα, a statement deemed true.
Theorem θεώρημα.
Hypothesis ὑπόθεσις, a supposition, a foundation, from ὑπό, τίθημι.
Identity idem, identidem, the same thing
Geometry γῆ, μετρέω, to measure land
Diameter διά μετρέω, to measure across

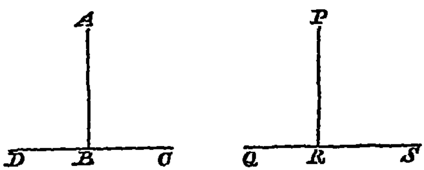
Plane. Planus, level.
Complement. Compleo, to fill up
Rectilinear Recta, linea, a straight line.
Polygon πολύς, γωνία, many angles
Diagonal διὰ γωνία, across from angle to angle
Perimeter περὶ μετρέω, to measure round
Triangle. Tres, angulus, with three angles
Quadrilateral Quadra (quater), latus, with four sides
Equilateral Æquus, latus, having equal sides
Pentagon. πέντε γωνία, with five angles
Postulate. Postulatum, a thing requested
Isosceles ἴσος σκέλος, ἰσοσκελής, having equal legs
Hypotenuse. ὑπὸ τείνουσα (γραμμή), the line stretching across.
Parallel παρὰ, ἄλληλα, alongside of one another.
Parallelogram παρὰ, ἄλληλα, γραμμή, made by lines alongside of one another.
Trapezium. τράπεζα, a table
Trapezoid τραπεζοειδής, like a table.
Orthogonal. ὀρθὸς, γωνία, having right angles.

SECTION I.

ANGLES AT A POINT.

THEOREM I.

All right angles are equal to one another.



Part En Let ABC, PRS be right angles;
it is required to prove that ABC is equal to PRS

Proof. If the point B were placed on the point R, and the line BC along the line RS,

then because the lines DC, QS are *straight*,
therefore the line BD would fall along the line RQ, (Ax. 2)
therefore the angle DBC coincides with, and is equal to, the angle QRS. (Ax. 1)

But by Def 14, the angle ABC is half the angle DBC, and the angle PRS is half the angle QRS,
and the halves of equals are equal,
therefore the right angle ABC is equal to the right angle PRS Q E D

COR 1 *At a given point in a given straight line there can be only one perpendicular drawn to that line*

COR. 2 *The complements of equal angles are equal*

COR 3 *The supplements of equal angles are equal*

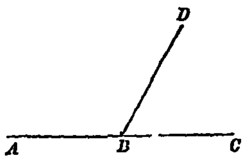
13 th.

THEOREM 2

*If a straight line stands upon another straight line it makes the adjacent angles together equal to two right angles**

Part En Let DB stand upon the straight line AC,

it is required to prove that the adjacent angles ABD, DBC are together equal to two right angles



Proof Because ABC is a *straight* line, (Hyp)
therefore the angle ABC is equal to two right angles, (Def 14)
but the angle ABC is, from the figure, made up of the angles ABD and DBC, (Def 12)
therefore the angles ABD and DBC are together equal to two right angles. Q E D.

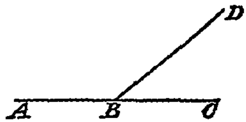
* Eucl. I. 13.

14th

Theorem 3.

If the adjacent angles made by one straight line with two others are together equal to two right angles, these two straight lines are in one straight line.*

Part. En. Let the adjacent angles DBA, DBC made by BD with the two straight lines BA, BC be together equal to two right angles;



it is required to prove that AB, BC are in one straight line.

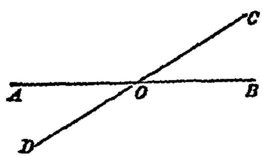
Proof. Because DBA and DBC are together equal to two right angles, (Hyp.)
and DBA and DBC, from the figure, make up the angle ABC;
therefore ABC is an angle of two right angles,
and therefore ABC is a straight line. (Def. 14.) Q. E. D.

15th

Theorem 4.

If two straight lines cut one another the vertically opposite angles will be equal to one another.

Part. En. Let the straight lines AOB, DOC cut one another, and let AOD, BOC be vertically opposite angles,



it is required to prove that the angle AOD is equal to the angle BOC.

Proof. Because AOB is a *straight* line; (Hyp.)
therefore the angles AOC and COB are together equal to two right angles. (Th. 2.)

* Eucl. I. 14.

And again because DOC is a *straight* line; (Hyp)
therefore the angles AOC and AOD are together equal to two right angles (Th 2)

Therefore the angles AOC and COB are equal to the angles AOC and AOD.

Take away the common angle AOC;
therefore the angle COB is equal to the angle AOD*.

Q E D

COR. *The sum of all the angles made by any number of lines taken consecutively which meet at a point is four right angles*†.

EXERCISES ON ANGLES

1 If two straight lines intersect at a point, and one of the four angles is a right angle, prove that the other three are right angles.

2 *Two* straight lines meet at a point Are the angles at that point together equal to four right angles?

3. If the four angles made by four straight lines which meet at a point are all right angles, prove that the four lines form two straight lines.

4. If five lines meet at a point and make equal angles with one another all round that point, each of the angles is four-fifths of a right angle.

5 Of two supplementary angles the greater is double of the less, find what fraction the less is of four right angles

6. Twelve lines meet at a point so as to form a regular twelve-rayed star. find the angle between consecutive rays

* Eucl I 15 † Eucl I 15 Cor.

7. If four straight lines OA, OB, OC, OD meet at a point, and $AOB = COD$, and $BOC = DOA$, prove that AOC, BOD are straight lines

8. Prove that the bisectors of adjacent supplementary angles are at right angles to one another.

9. Find the angle between the bisectors of adjacent complementary angles.

10. Prove that the bisectors of the four angles which one straight line makes with another form two straight lines at right angles to one another.

11. If four lines AO, BO, CO, DO meet at a point O, and the angles AOB, COD are given equal, and also AO, CO are given as being in the same straight line; prove that BO and DO, if on opposite sides of AOC, are also in the same straight line.

12. If the corner of the page of a book be folded down so as to form an oblique crease, prove that the bisector of the angle between the parts of the edge that meet at the crease will be at right angles to the crease.

QUESTIONS ON SECTION I.

1 What is meant by the Elements of Plane Geometry?

2 Explain the terms *axiom*, *theorem*, *converse*, *contrapositive*, giving examples of each.

3. State the Geometrical Axioms

4. What is meant by the axiom of the rule of Identity?

5 State the fact that "all geese have two legs" in the form of a theorem, with hypothesis and conclusion; and write down its obverse, converse, and contrapositive theorems.

6 Define a plane surface, and give the test by which a surface is ascertained to be or not to be plane

7 On what does the magnitude of an angle depend? Shew that its magnitude does not depend on the length of the arms

8 What is meant by saying that two points *determine* a straight line?

9 What are *adjacent* angles, *supplementary* angles, *reflex* angles?

10 Shew how to find the sum and difference of two straight lines and prove that their sum and difference together are double of the greater of the two straight lines

11. Given the sum and difference of two straight lines, find the lengths of the straight lines

12. Enunciate and prove the obverse and converse of Theorem 4.

SECTION II.

TRIANGLES.

Def. 30. An *isosceles* triangle is that which has two sides equal.

Def. 31. A *right-angled* triangle is that which has one of its angles a right angle. An *obtuse-angled* triangle is that which has one of its angles an obtuse angle. All other triangles are called *acute-angled triangles.*

Def. 32. A triangle is sometimes regarded as standing on a selected side which is then called its *base*, and the intersection of the other two sides is called the *vertex* When two of the sides of a triangle have been mentioned, the remaining side is often called the *base.*

Def. 33. The side of a right-angled triangle which is opposite to the right angle is called the *hypotenuse*

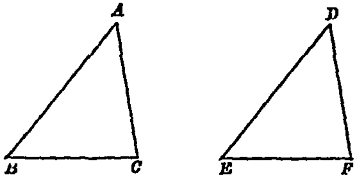
Def 34. Figures that may be made by superposition to coincide with one another are said to be *identically* equal; and every part of one being equal to a corresponding part of the other, they are said to be equal in all respects

44

THEOREM 5.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by these sides equal, then the triangles are identically equal, and of the angles those are equal which are opposite to the equal sides.

Part En Let the two triangles BAC, EDF have two sides of the one equal to two sides of the other, each to each, and likewise the included angles equal, viz



$$\left.\begin{array}{r} BA = ED, \\ AC = DF, \\ \text{and the included angle } BAC = \text{the included angle } EDF, \end{array}\right\}$$

it is required to prove that the triangles are equal in all respects,

viz the base BC equal to the base EF, and the angle B to the angle E, and the angle C to the angle F, and the area ABC to the area DEF

Proof If the point A be placed on the point D,
and the line AB were placed along DE,
then because the angle BAC = the angle EDF, (Hyp)
therefore the line AC would lie along DF.

And because $AB = DE$, (Hyp)
therefore the point B would coincide with the point E

And because $AC = DF$, (Hyp)
therefore the point C would coincide with the point F

Therefore BC would coincide with EF, (Ax. 2)
and therefore $BC = EF$ (Ax. 1)
and the angles B and C respectively coincide with and are equal to the angles E and F,
and the area of the triangle BAC coincides with and is equal to the area of the triangle EDF*. Q. E. D.

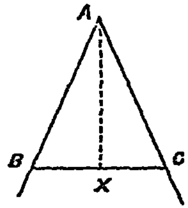
* Eucl I 4

THEOREM 6.

*The angles at the base of an isosceles triangle are equal to one another**.

Part. En. Let ABC be an isosceles triangle, having the side AB equal to the side AC;

it is required to prove that the angle B is equal to the angle C.



Proof. Let AX be the bisector of the angle BAC, (Ax. 4)

meeting the base BC in X.

Then in the triangles BAX, CAX we have

$BA = AC$, (Hyp.)

AX common,

and the included angle $BAX =$ the included angle CAX. (Hyp.)

Therefore the triangles are equal in all respects, (Th. 5)

that is, the angle at $B =$ the angle at C. Q. E. D.

COR. 1. *If the equal sides be produced the angles on the other side of the base will be equal.*

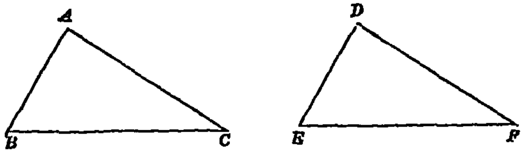
COR. 2. *If a triangle is equilateral it is also equiangular.*

* Eucl. I. 5.

THEOREM 7.

If two triangles have one side of the one equal to one side of the other, and the angles at the extremities of those sides equal, each to each, then the triangles are equal in all respects, those sides being equal which are opposite to the equal angles.*

Part En Let the triangles ABC, DEF have



$$\left.\begin{aligned} BC &= EF, \\ \text{the angle } B &= \text{the angle } E, \\ \text{and the angle } C &= \text{the angle } F, \end{aligned}\right\}$$

it is required to prove that the triangles are equal in all respects

Proof For if the point B were placed on the point E, and the line BC along the line EF,
then because $BC = EF$, (Hyp)
therefore the point C would fall on F

And because the angle B = the angle E, (Hyp)
therefore the line BA would fall along the line ED.

And because the angle C = the angle F, (Hyp)
therefore the line CA would fall along the line FD:
therefore the point A would fall on the point D,
since two straight lines can intersect in one point only; (Ax. 2.)
and therefore the triangles coincide and are equal in all respects, AB being equal to DE, AC to DF, and the

* Eucl I 26. Part I

angle A to the angle D, and the area ABC to the area DEF. Q. E. D.

Theorem 8.

If the angles at the base of a triangle are equal to one another, the triangle is isosceles.*

Part. En. Let the two angles B and C of the triangle ABC be equal; it is required to prove that $AB = AC$.

Proof. If the triangle were taken up and reversed and replaced, so that the point C fell where B was, and the line CB along the line BC, then B would fall where C was.

And because the angle C = the angle B, (Hyp.) the line CA would lie along BA, and BA along CA; therefore the point A would coincide with its former position, and the lines AC, AB would coincide with the lines AB, AC.

Therefore $AB = AC$. Q. E. D.

Cor. *If a triangle is equiangular, it is also equilateral.*

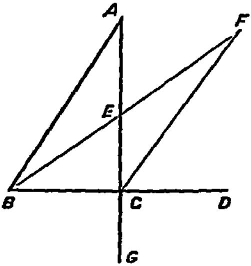
Theorem 9.

If any side of a triangle be produced, the exterior angle will be greater than either of the interior and opposite angles.

Part. En. Let ABC be a triangle, and let one of its sides BC be produced to D;
it is required to prove that the exterior angle ACD is greater than either of the interior and opposite angles CAB or ABC.

* Euclid, I. 6.

Proof Firstly to prove that ACD is greater than BAC
Let AC be bisected in E (Ax 3)



Join BE, and produce it to F, making $EF = EB$ And join FC.

Then in the triangles AEB, CEF we have

$AE = EC$, (Constr)
$BE = EF$, (Constr)
and the contained angles AEB, CEF are equal, (Th 4.)
therefore the triangles are equal in all respects, (Th. 5)
and therefore the angle EAB = the angle ECF.

But the angle ECD is greater than ECF,
therefore the angle ECD is also greater than EAB.

Again, if AC is produced to G, and BC is bisected, it may be similarly shewn that BCG is greater than ABC

but BCG is equal to ACD,

therefore ACD is also greater than ABC,

that is, the exterior angle ACD is greater than either CAB or ABC*. Q E D

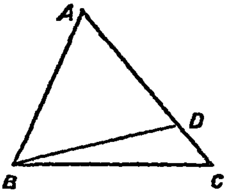
18th ·

THEOREM 10

The greater side of every triangle has the greater angle opposite to it.

* Euclid, I 16.

Part. En Let ABC be a triangle having AC greater than AB;



it is required to prove that the angle ABC is greater than the angle ACB.

Proof. From AC cut off $AD = AB$; and join DB.

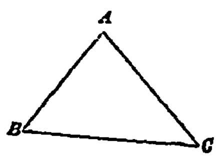
Because $AD = AB$; (Constr.)
therefore the angle $ABD =$ the angle ADB. (Th. 6.)

But because ADB is the exterior angle of the triangle BDC,
therefore the angle ADB is greater than the angle ACB, (Th. 9)
therefore also the angle ABD is greater than the angle ACB.
much more then is the angle ABC greater than the angle ACB*. Q. E. D.

THEOREM 11.

The greater angle of every triangle has the greater side opposite to it.

Part En. Let ABC be a triangle in which the angle B is greater than the angle C;



it is required to prove that the side AC is greater than the side AB.

Proof. For AC must be either equal to AB, or less than AB, or greater than AB.

* Euclid, I. 18.

But AC is not equal to AB, for then the angle B would be equal to the angle C (Th 6)

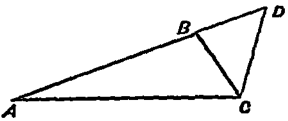
Nor is AC less than AB,
for then the angle B would be less than the angle C. (Th 10)

Therefore AC is greater than AB*. Q E D

THEOREM 12.

Any two sides of a triangle are together greater than the third side.

Part En Let ABC be a triangle, it is required to prove that AB and BC are together greater than AC



Proof Produce AB to D, making $BD = BC$,
join DC

Then because $BD = BC$, (Constr)
therefore the angle BCD = the angle BDC (Th 6)

But the angle ACD is greater than the angle BCD,
therefore the angle ACD is greater than the angle ADC,
and therefore AD is greater than AC (Th 11)

But AD is equal to AB and BC together,
therefore AB and BC are together greater than AC† Q.E D

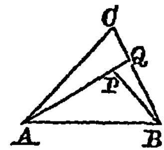
COR *The difference of any two sides of a triangle is less than the third side*

* Euclid, I 19 † Euclid, I 20.

THEOREM 13.

If from the ends of the side of a triangle two straight lines be drawn to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

Part. En Let ACB be a triangle, and from the ends of the side AB let two straight lines AP, BP be drawn to a point P within the triangle; it is required to prove that AP and PB are less than AC and CB, but the angle APB greater than the angle ACB.



Proof. Produce AP to meet BC in Q.

Because any two sides of a triangle are together greater than the third side, (Th 12)

therefore AC and CQ are greater than AQ;

add to each QB,

therefore AC and CB are greater than AQ and QB.

Again, because PQ and QB are greater than PB; (Th. 12.)

add to each AP;

therefore AQ and QB are greater than AP and PB;

but AC and CB are greater than AQ and QB;

much more then are AC and CB greater than AP and PB.

Again, because APB is the exterior angle of the triangle PQB:

therefore the angle APB is greater than the angle PQB; (Th. 9)

and because PQB is the exterior angle of the triangle ACQ;

therefore the angle PQB is greater than the angle ACQ; (Th. 9)

but the angle APB is greater than the angle PQB;
much more then is the angle APB greater than the angle ACB*.

Q E D

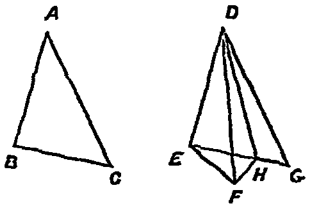
THEOREM 14

If two triangles have two sides of the one equal to two sides of the other, each to each, but the included angles unequal, their bases are unequal, the base of that which has the greater angle being greater than the base of the other †.

Part En Let ABC, DEF be two triangles, having

$$AB = DE,$$
$$AC = DF,$$

but the included angle BAC greater than the included angle EDF,



it is required to prove that the base BC is greater than the base EF.

Proof. Place the point A on D, and AB along DE;
then because $AB = DE$, (Hyp)
therefore the point B will fall on the point E,
and because the angle BAC is greater than the angle EDF,
the line AC will fall outside DF, as DG, (Hyp)
and BC will fall as EG

Let DH be the bisector of the angle FDG, (Ax 4)
meeting EG in H

* Euclid, I 21 † Euclid, I 24.

Join FH.

Then because in the triangles FDH, GDH, we have

$$\left.\begin{array}{ll} FD = GD, & \text{(Hyp.)} \\ DH \text{ common,} & \\ \text{and the included angle } FDH = \text{the included angle } GDH, & \end{array}\right\}$$

(Constr.)

therefore $HF = HG$; (Th. 5.)

and therefore EH and HF together are equal to EG.

But EH and HF together are greater than EF, (Th. 12.)

therefore EG or BC is greater than EF. Q. E. D.

THEOREM 15.

If two triangles have the three sides of the one equal to the three sides of the other, each to each, then the triangles are identically equal, and of the angles those are equal which are opposite to equal sides.*

Part. En. Let ABC, DEF be two triangles

which have

$$\left.\begin{array}{l} AB = DE, \\ BC = EF, \\ CA = FD, \end{array}\right\}$$

then shall the triangles be equal in all respects.

Proof. The angle BAC must be either equal to EDF, or greater than EDF, or less than EDF.

But BAC is not greater than EDF,

for then the base BC would be greater than the base EF. (Th. ~~14~~.)

Nor is BAC less than EDF,

for then the base BC would be less than the base EF. (Th. 14.)

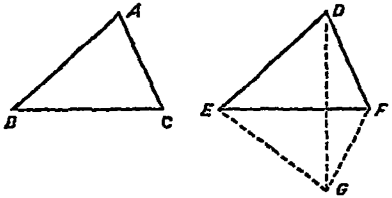
therefore the angle BAC is equal to the angle EDF.

Q. E. D.

* Eucl. I. 8.

Alternative Proof

If the point B were placed on E, and BC were placed along EF, then because $BC=EF$ (Hyp), therefore the



point C would fall on F and let the triangles BAC, EDF fall on opposite sides of EF, BA, AC falling as EG, GF, and the angle BAC as EGF. Join DG.

Then because $EG=ED$, (Hyp)
therefore the angle EDG = the angle EGD; (Th 6)
and because $FG=FD$, (Hyp)
therefore the angle FDG = the angle FGD. (Th 6)
therefore the whole angle EDF = the whole angle EGF,
but $EGF=BAC$, (Constr)
therefore $EDF=BAC$,
and therefore the triangles BAC, EDF are equal in all respects (Th 5) Q. E. D.

NOTE The student should examine for himself the cases in which DG passes through an extremity of the base, and passes outside the base.

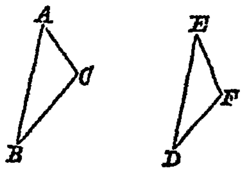
THEOREM 16

25T.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one is greater than the base of the other, then the angle contained by the sides of that which has the greater base is greater than the angle contained by the sides of the other

Part. En. Let BAC, DEF be two triangles which have

$$\left.\begin{aligned} BA &= DE, \\ AC &= EF, \\ \text{but the base } BC &\text{ greater than the base } DF; \end{aligned}\right\}$$



it is required to prove that the angle BAC is greater than the angle DEF.

Proof. For the angle BAC must either be equal to the angle DEF, or less than the angle DEF, or greater than the angle DEF.

But the angle BAC is not equal to the angle DEF, for then the base BC would be equal to the base DF, but it is not. (Th 5)

Nor is the angle BAC less than the angle DEF, for then the base BC would be less than the base DF, but it is not. (Th. 14.)

Therefore the angle BAC must be greater than the angle DEF*. Q. E. D.

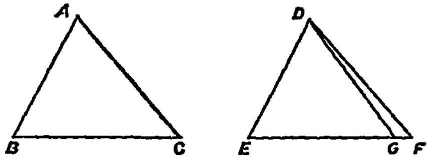
THEOREM 17.

If two triangles have two angles of the one equal to two angles of the other, each to each, and have the sides opposite to one of the equal angles in each equal, then the triangles are equal in all respects, those sides being equal which are opposite to the equal angles.

* Euclid I. 25.

Part En Let the two triangles ABC, DEF have the two angles

$$\left.\begin{aligned} ABC &= DEF, \\ ACB &= DFE, \\ \text{and the side } AB &= \text{the side } DE, \end{aligned}\right\}$$



it is required to prove the triangles are equal in all respects

Proof Let the point A be placed on the point D, and AB along DE,

then because $AB = DE$, (Hyp)

therefore the point B will fall on the point E.

And because the angle ABC is equal to the angle DEF, (Hyp)

therefore the line BC will lie along EF

And the point C will fall on F, for if it fell otherwise as G,

then, since the angle ACB is equal to the angle EFD, (Hyp)

the angle EGD would be equal to the angle EFD, the exterior angle equal to the interior and opposite, which is impossible, (Th 9)

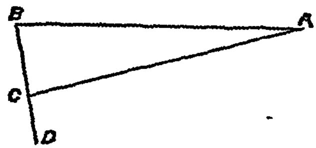
therefore the triangles would coincide and are equal in all respects, AC being equal to DF, BC to EF, and the angle BAC to the angle EDF*. Q E D

* Eucl I 26. Part 2.

17th

THEOREM 18.

Any two angles of a triangle are together less than two right angles.



Part. En. Let ABC be a triangle, it is required to prove that any two of its angles ABC and ACB are together less than two right angles.

Proof. Produce the side BC to D.

Then because the exterior angle ACD is greater than the interior and opposite angle ABC; (Th. 9.)
add to each the angle ACB;

therefore the two angles ACD and ACB are greater than the two ABC and ACB.

But ACD and ACB are together equal to two right angles; (Th. 2.)

therefore ABC and ACB are together less than two right angles*. Q. E. D.

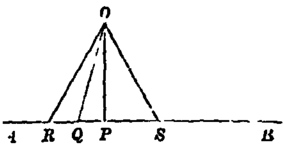
COR. 1. *If one angle of a triangle is right or obtuse, the others are acute.*

COR. 2. *From a given point outside a given straight line, only one perpendicular can be drawn to that straight line.*

* Euclid, I. 17.

THEOREM 19

Of all the straight lines that can be drawn from a given point to meet a given straight line, the perpendicular is the shortest, and of the others, those making equal angles with the perpendicular are equal, and that which makes a greater angle with the perpendicular is greater than that which makes a less



Part En Let O be the given point, and AB the given straight line, and let OP be the perpendicular, OQ an oblique,

it is required to prove first that OP is less than OQ

Proof Since any two angles of a triangle are together less than two right angles, (Th 18)

therefore OPQ and OQP are together less than two right angles

but OPQ is a right angle, (Hyp)

therefore OQP is less than a right angle

And in the triangle OQP, since the angle OPQ is greater than the angle OQP,

therefore OQ is greater than OP. (Th 11)

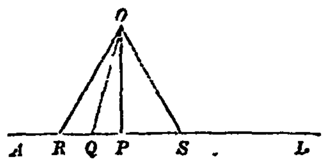
Again, let OS, OR be obliques making equal angles with the perpendicular OP;

it is required to prove that $OR = OS$

Because in the triangles POR, POS

the angle $OPR = OPS$, being right angles, (Hyp)
and the angle $POR = POS$, (Hyp)
and PO is common;

therefore the triangles are equal in all respects, (Th 7)
and therefore $OR = OS$



Lastly, let OR make a greater angle with the perpendicular than OQ;
it is required to prove that OR is greater than OQ

Because OQR is the exterior angle of the triangle OQP,
therefore OQR is greater than OPQ; (Th. 9)
but OPQ is a right angle; (Hyp)
therefore OQR is an obtuse angle;
therefore ORQ is an acute angle, and less than OQR; (Th. 18)
and therefore OR is greater than OQ. Q E D

Cor. *Not more than two equal straight lines can be drawn from a given point to a given straight line.*

Theorem 20.

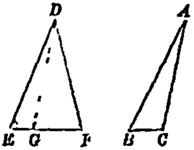
If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles opposite to two equal sides equal, the angles opposite to the other two equal sides are either equal or supplementary, and in the former case the triangles are equal in all respects

Part En Let ABC, DEF be the two triangles, having the sides BA, AC equal to the sides ED, DF respectively, and having also the angle B = the angle E,

it is required to prove that the angle C is equal or supplementary to the angle F, and that when the angle C is equal to the angle F, the triangles are equal in all respects

Fig 1

Fig 2



Proof The contained angle A must be either equal or unequal to the contained angle D.

If $A = D$, as in Fig 1, then, by Theorem 5, the triangles are equal in all respects, and the angle C is *equal* to the angle F.

If A is not equal to D, as in Fig 2, let the point A be placed on D, and AB along DE,

then, because $AB = DE$, (Hyp)

the point B will coincide with the point E,

and because the angle at B = the angle at E, (Hyp)

therefore the line BC will lie along EF,

and the point C will fall on EF as G.

and because $AC = DF$, (Hyp)

therefore $DG = DF$,

and therefore the angle DFE = the angle DGF, (Th 6)

but DGF is supplementary to DGE, that is, to ACB, and therefore the angle F is *supplementary* to the angle C

Q E D.

Cor. *Hence the triangles are equal in all respects—*

(1) *If the two angles given equal are right angles or obtuse angles.*

For then the remaining angles must be acute, and therefore cannot be supplementary, and must therefore be equal by the Theorem, and therefore the triangles must be equal in all respects.

(2) *If the angles opposite to the other two equal sides are both acute, or both obtuse, or if one of them is a right angle*

(3) *If the side opposite the given angle in each triangle is not less than the other given side.*

For then the given angles must be the greater of the two, and therefore the remaining angles must be both acute, and therefore cannot be supplementary, and must therefore be equal, by the Theorem, and therefore the triangles must be equal in all respects

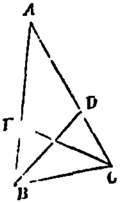
Exercises on Theorems of Equality.

The general method to be adopted in the solution of theorems of equality is the following Examine fully the statement of the question, see what is included among the *data:* what lines and angles are *given* equal by *hypothesis.* Then see what is required to be proved, what lines or angles have to be proved to be equal. It may follow from the properties proved of a single triangle; or it may depend on the equality of a pair of triangles In the latter case examine the triangles of which they form corresponding parts, and see whether the data are sufficient to prove *these* triangles equal If the data are sufficient, the solution is effected by comparing the triangles, and shewing the required equality of the lines and angles; if not, the data must be used to establish results, which in their turn can be used to establish the conclusion required.

The beginner will do well to arrange his proofs in the manner shewn in the example, giving references in the margin.

EXERCISES

Ex (1) *The lines which bisect the angles at the base of an isosceles triangle, and meet the opposite sides, are equal*



Let ABC be an isosceles triangle.

Data $AB=AC$, and the angles at B and C bisected by BD, CE

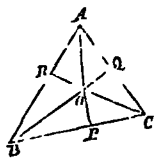
Proof In the triangles ACE, ABD we have

$AC=AB$, (Hyp)
angle at A common,
and angle ACE=angle ABD (Hyp and Th 6)

Therefore the base CE=the base BD (Th 7) Q E D

Ex (2) *The bisectors of the three angles of a triangle will meet in one point*

Let ABC be a triangle, and let the bisectors of the angles ABC, ACB be BO, CO, meeting in O, then the Theorem will be proved if we can shew that AO is the bisector of the angle BAC



Let perpendiculars OP, OQ, OR be drawn to the three sides BC, CA, AB

Proof In the triangles OQC, OPC we have

$OQC=OPC$, (Constr)
$OCQ=OCP$, (Hyp)
OC common

Therefore $OQ=OP$ by Theorem 17

Similarly from the triangles OPB, ORB, it follows that $OP=OR$, therefore $OR=OQ$,

and therefore the right-angled triangles OQA, ORA have the hypotenuse and one side of the one equal to the hypotenuse and one side of the other, and are therefore equal in all respects by Theorem 20, Cor 1

Therefore the angle OAQ=the angle OAR, that is, OA is the bisector of the angle BAC

Exercises for Solution.

1. OA and OB are any two equal lines, and AB is joined; shew that AB makes equal angles with OA and OB.

2. If the bisectors of the equal angles B, C of an isosceles triangle meet in O, shew that OBC is also an isosceles triangle.

3. The line drawn to bisect the vertical angle of an isosceles triangle will also bisect the base, and be perpendicular to it.

4. The lines joining the middle points of the sides of an isosceles triangle to the opposite extremities of the base will be equal to one another.

5. The line drawn from the vertex of an isosceles triangle to bisect the base will cut it at right angles, and bisect the vertical angle.

6. Prove that the lines which bisect the sides of a triangle and are perpendicular to them meet in one point.

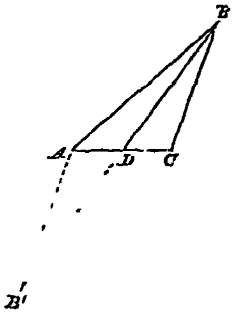
7. The perpendiculars let fall from the extremities of the base of an isosceles triangle upon the opposite sides will be equal, and will make equal angles with the base.

8. The perpendicular let fall from the vertex of an isosceles triangle to the base, will bisect the base and the vertical angle.

9. If two exterior angles of a triangle be bisected by straight lines which meet in O, prove that the perpendiculars from O on the sides or sides produced of the triangle are equal to one another.

EXERCISES ON THEOREMS OF INEQUALITY

The line that joins the vertex to the middle point of the base of a triangle is less than half the sum of the two sides



Let D be the middle point of AC, then is BD less than half the sum of AB, BC.

Proof. Produce BD to B', making $DB'=DB$ Join AB'.

Then since the two triangles BDC, $B'DA$ have two sides BD, DC and the included angle BDC of the one respectively equal to the two sides $B'D$, DA and the included angle $B'DA$ of the other, therefore (Theorem 5) the base BC = the base AB',

but $$B'A+AB>B'B, \qquad \text{(Th 12)}$$

$$\therefore AB+BC>B'B, \text{ which is twice } BD,$$

that is, BD is less than half the sum of BC and BA

EXERCISES FOR SOLUTION.

1. Prove that any one side of a four-sided figure is less than the sum of the other three sides

2 Prove that the sum of the lines which join the opposite angles of any four-sided figure is together greater than the sum of either pair of opposite sides of the figure.

3 Prove that the sum of the diagonals of a quadrilateral figure is less than the sum of the four lines which can be drawn to the angles from any other point than the intersection of the diagonals

4. O is any point within the triangle ABC; prove that $OA + OB + OC$ are less than the sum and greater than half the sum of $AB + BC + CA$.

5. Prove that the sum of the four sides of a quadrilateral figure is greater than the sum and less than twice the sum of the diagonals.

6. If ABC is a triangle in which AB is greater than AC, and D is the middle point of BC, and AD is joined, prove that the angle ADB is an obtuse angle.

7. Prove that the sum of the three sides of a triangle is greater than the sum of the three medians.

NOTE. *The median of a triangle is the line that joins any angle to the middle point of the opposite side.*

8. Prove that the sum of the three medians of a triangle is greater than half the sum of the sides.

QUESTIONS ON SECTION II.

1. Give the meaning and derivation of the words *triangle*, *perpendicular*, *isosceles*, *equilateral*, *hypotenuse*, *median*.

2. If a triangle is isosceles, the angles at its base will be equal. Enunciate the obverse, converse and contrapositive theorems.

3. Apply Theorem 7 to find the height of a tower.

4. Prove Theorem 6 in the manner of Theorem 8.

5. Why cannot Theorem 15 be proved in the same manner as Theorem 5?

6. Prove that only one perpendicular can be drawn from a given point to a given straight line.

7. Prove fully the corollary to Theorem 19.

8. Enumerate the five cases in which the equality of three parts in a pair of triangles involves the equality in all respects.

9 Mention cases, and draw the figures, in which two triangles are equal in three respects but not in all

10 Prove fully the corollaries to Theorem 20

11. Prove Theorem 19 by conceiving the figure to be folded down over the line AB, O falling on a point O', and RO, QO, PO, on RO', QO', PO', and using Theorems 12 and 13

12. In Theorem 9, prove fully that ACD is greater than ABC

13 Why is it necessary, in the enunciation of Theorem 9, to say interior *and opposite* angles?

14 What is the magnitude indirectly measured in Theorem 9?

15 Enunciate Theorem 10 formally Is it merely the obverse of Theorem 6, or does it contain an additional geometrical fact?

16 Prove Theorem 10 by reversal and superposition, using Theorem 9

17 Shew how Theorem 12 depends ultimately on the Axioms

18 Which Theorem in this Section proves that as you increase the angle between the legs of a pair of compasses you also increase the distance between their points?

19 Shew the relation of Theorems 14, 15 and 16 to Theorem 5

20 Enunciate the contra positives of Theorems 9 and 18

SECTION III.

Parallels and Parallelograms

Def. 35. *Parallel* straight lines are such as are in the same plane and being produced to any length both ways do not meet.

Axiom 5. Two straight lines that intersect one another cannot both be parallel to the same straight line.

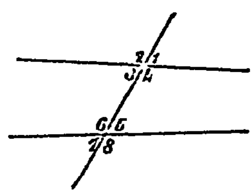
Def. 36. A *trapezium* is a quadrilateral that has only one pair of opposite sides parallel.

This figure is sometimes called a *trapezoid.*

Def. 37. A *parallelogram* is a quadrilateral whose opposite sides are parallel.

Def. 38. When a straight line intersects two other straight lines it makes with them eight angles, which have received special names in relation to one another.

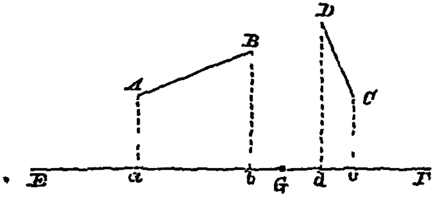
Thus in the figure 1, 2, 7, 8 are called *exterior* angles, and 3, 4, 5, 6, *interior* angles; again, 4 and 6, 3 and 5, are called *alternate* angles; lastly, 1 and 5, 2 and 6, 3 and 7, 4 and 8, are called *corresponding* angles.



Def. 39. The *orthogonal projection of one straight line on another straight line* is the portion of the latter intercepted

between perpendiculars let fall on it from the extremities of the former.

Thus the projections of AB, CD on EF are the lines ab, cd respectively



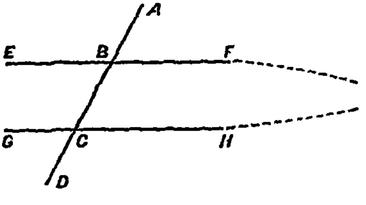
It is clear that the line EF must be supposed indefinitely long There could be no projection of AB on the terminated line GF.

27th

THEOREM 21

If one straight line intersects two other straight lines so as to make the alternate angles equal, the straight lines are parallel

Part. En Let $ABCD$ intersect EF and GH, and make the angle EBC equal to its alternate angle BCH, it is required to prove that EF is parallel to GH



Proof For if EF and GH meet towards F, H, they would form a triangle with BC,

and EBC would be its exterior angle, and therefore greater than the interior and opposite angle BCH. (Th. 9.)

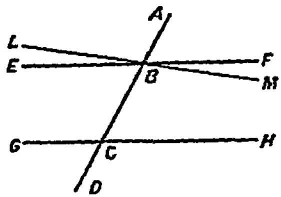
But EBC is equal to BCH, (Hyp.)
therefore EF and GH do not meet towards F, H.

Similarly they do not meet towards E, G,
that is, EF is parallel to GF*. (Def. 35.) Q. E. D.

THEOREM 22.

If two straight lines are parallel, and are intersected by a third straight line, the alternate angles are equal†.

Part. En. Let EF and GH be parallel straight lines, and let $ABCD$ intersect them;
it is required to prove that the alternate angles EBC, BCH are equal.



Proof. For if EBC were not equal to BCH,
let some other line LBM be drawn through B making the angle LBC equal to the alternate angle BCH,
then LM would be parallel to GH. (Th. 21.)

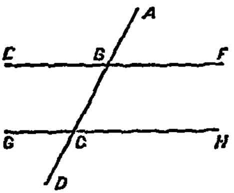
But EF is parallel to GH; (Hyp.)
that is, two intersecting lines LM, EF would be both parallel to GH; which is impossible. (Ax. 5.)

Therefore EBC is equal to BCH, that is, the alternate angles are equal. Q. E. D.

* Euclid, I. 27. † Euclid, I. 29.

THEOREM 23

If a straight line intersects two other straight lines and makes either a pair of alternate angles equal, or a pair of corresponding angles equal, or a pair of interior angles on the same side supplementary, then, in each case, the two pairs of alternate angles are equal, and the four pairs of corresponding angles are equal, and the two pairs of interior angles on the same side are supplementary



Part En Let the straight line $ABCD$ intersect the two straight lines EF, GH, and make the alternate angles EBC, BCH equal, then will the other alternate angles FBC, BCG be equal, and the four pairs of corresponding angles be equal, and the two interior angles on the same side be supplementary

Because $\qquad EBC = BCH,\qquad$ (Hyp)

and $EBC = ABF$ being vertically opposite angles, (Th 4)

therefore $\qquad ABF = BCH,$

therefore also their supplements, the angles ABE and BCG are equal

Therefore also the angles which are respectively vertically opposite to these angles are equal,

that is, the angle $\qquad CBF = DCH,$

and $\qquad EBC = GCD$

Again, because the angle EBC = the alternate angle BCH add to each the angle CBF,
therefore the two angles EBC, CBF are equal to the two CBF, BCH;
but the two EBC, CBF are together equal to two right angles,
therefore the two CBF, BCH are together equal to two right angles.

And in the same way it may be shewn that if two corresponding angles are given equal, or if two interior angles on the same side are supplementary, then the alternate angles will be equal.

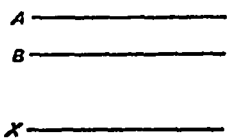
Cor. *Hence if two parallel straight lines are intersected by a third straight line, the corresponding angles are equal, and the interior angles on the same side are supplementary; and conversely.*

Theorem 24.

Straight lines which are parallel to the same straight line are parallel to one another.*

Part En. Let A and B be each of them parallel to X, it is required to prove that A is parallel to B.

Proof. If A intersected B, then two intersecting lines,



A, B would each be parallel to a third line X, which is impossible, by Axiom 5.

Therefore A does not intersect B,
that is, A is parallel to B. Q. E. D.

* Euclid, I. 30.

REMARKS.

It must be observed that two parallels, and a straight line intersecting them, are a special case of the triangle, the vertex, or intersection of two of the lines, being removed to an infinite distance In Th 9 it was proved that the exterior angle of a *triangle* is *greater* than the interior and opposite angle from which the contra-positive theorem (Th 21) logically follows, that if the exterior angle is *equal* to the interior and opposite angle, the lines do *not* form a triangle.

Theorem 22 is in fact proved by the rule of identity (p 4)

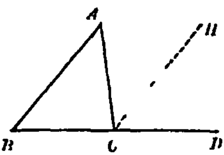
Since there is only one straight line through B that makes the alternate angles equal,

and only one straight line through B that is parallel to GH, (Ax 5)
and the line that makes the alternate angles equal is the parallel, (Th 21)
therefore the parallel makes the alternate angles equal

32nd

THEOREM 25

If one side of a triangle be produced the exterior angle will be equal to the two interior and opposite angles, and the three interior angles of a triangle are together equal to two right angles



Let one side BC of the triangle ABC be produced to D. then shall the angle ACD = the sum of the angles ABC, CAB, and the three angles ABC, BCA, CAB shall be together equal to two right angles

Proof For if through C a line CH were drawn parallel to BA,

the angle HCD = the corresponding angle ABC, (Th 22)
and the angle ACH = the alternate angle BAC; (Th 22)
the whole angle ACD = the two angles $ABC + BAC$.

Again, if ACB be added to these,
the two angles $ACD + ACB =$ the three angles $ABC + BCA + CAB$.

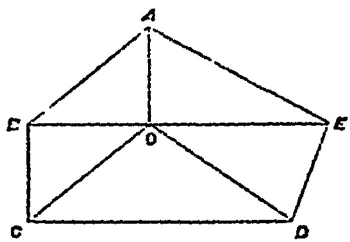
But $ACD + ACB =$ two right angles; (Th. 4)
therefore $ABC + BCA + CAB =$ two right angles*. Q. E. D.

COR. *In a right-angled triangle the two acute angles together make up one right angle.*

THEOREM 26.

The interior angles of any polygon are together less than twice as many right angles as the figure has sides by four right angles.

Part. En. Let $ABCDE$ be any polygon; it is required to prove that its interior angles are together less than twice as many right angles as the figure has sides by four right angles.



Proof. Take any point O within the polygon; and join OA, OB, OC, OD, OE.

Then there are as many triangles having O as a common vertex as the figure has sides.

And, since the interior angles of a triangle are equal to two right angles, (Th. 25)

* Euclid, I. 32.

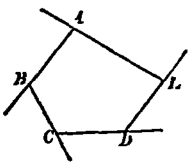
therefore all the angles of all the triangles are equal to twice as many right angles as the figure has sides

But all the angles of all the triangles make up all the angles of the polygon together with the angles at *O*, which are equal to four right angles, (Th 4. Cor)
therefore all the angles of the polygon, together with four right angles, are equal to twice as many right angles as the figure has sides,

that is, all the angles of the polygon are together less than twice as many right angles as the figure has sides by four right angles*.

COR. *The exterior angles of any convex polygon are together equal to four right angles*

Let *ABCDE* be a convex polygon having all its sides *AB*, *BC*, *CD*, *DE*, *EA* produced,



it is required to prove that the sum of its exterior angles is equal to four right angles.

Proof Each interior angle together with its adjacent exterior angle are equal to two right angles, (Th 2)
therefore all the interior angles together with all the exterior angles are equal to twice as many right angles as the figure has sides,

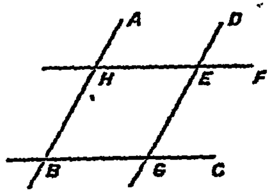
* Euclid, I 32 Cor I

but all the interior angles, together with four right angles, are equal to twice as many right angles as the figure has sides; (Th. 26.)

therefore all the exterior angles are equal to four right angles*. Q. E. D.

Theorem 27.

The adjoining angles of a parallelogram are supplementary and the opposite angles are equal.



Part En. Let $HBGE$ be a parallelogram,
that is, let HE, EG be respectively parallel to BG, BH; (Def. 37.)

it is required to prove that its adjoining angles EHB, HBG are supplementary, and its opposite angles HBG, HEG are equal.

Proof. Because HE is parallel to BG, (Hyp.)
and HB meets them,

therefore HBG is supplementary to EHB. (Th. 23 Cor.)

And because HB is parallel to EG, (Hyp.)

and HE meets them,

therefore HEG is supplementary to EHB; (Th. 23 Cor.)

but HBG is also supplementary to EHB,

therefore HEG is equal to HBG. (Th. 1. Cor. 3) Q. E. D.

* Euclid, I. 32. Cor. 2.

COR. 1. *Hence if one of the angles of a parallelogram is a right angle, all its angles are right angles*

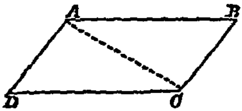
COR. 2. *If two straight lines are respectively parallel to two other straight lines they will include equal angles towards the same parts.*

Def 40. A right-angled parallelogram is called a *rectangle*

THEOREM 28.

34 *The opposite sides of a parallelogram are equal to one another, and the diagonal bisects it*

Part. En Let $ABCD$ be a parallelogram, that is, let AB be parallel to CD, and AD to BC, it is required to prove that AB is equal to DC, and AD to BC.



Proof. Join AC.

Then because AB is parallel to DC, and AC meets them; (Hyp)

therefore the angle BAC is equal to the alternate angle ACD. (Th. 22)

And because AD is parallel to BC, (Hyp)

therefore the angle BCA is equal to the alternate angle CAD: (Th. 22.)

therefore in the triangles BAC, DCA we have

the angle BAC = the angle DCA,
and the angle BCA = the angle DAC;
and the side AC adjacent to the equal angles common,

therefore the triangles are equal in all respects, (Th. 7.)

that is, AB is equal to DC, AD to BC, and the area ABC to the area ADC*. Q E. D

Cor. *Hence if the adjacent sides of a parallelogram are equal, all its sides are equal.*

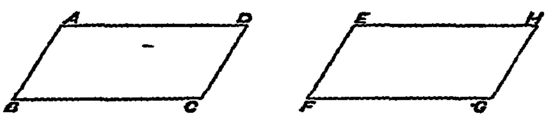
Def. 41. A parallelogram all whose sides are equal is called a *rhombus*

Def. 42. A *square* is a rectangle that has all its sides equal

Theorem 29

If two parallelograms have two adjacent sides of the one respectively equal to two adjacent sides of the other, and likewise an angle of the one equal to an angle of the other; the parallelograms are identically equal.

Part. En. Let $ABCD$, $EFGH$ be two parallelograms which have two adjoining sides AB, BC of the one equal respectively to two adjoining sides EF, FG of the other, and have likewise the included angles B and F equal,



it is required to prove that the parallelograms are identically equal.

Proof. For if the point B were placed on F, and the line BC along the line FG;

then because $BC = FG$, (Hyp)

therefore the point C will fall on G;

and because the angle ABC = the angle EFG, (Hyp)

* Euclid, I 34.

therefore BA will fall along FE,
and because $BA = FE$, (Hyp)
therefore the point A will fall on E

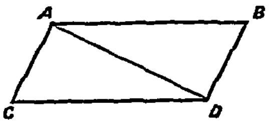
And because AD is parallel to BC, (Hyp)
therefore AD will fall along EH, (Ax 5)
and similarly CD will fall along GH,
and therefore the point D will fall on the point H,
that is, the parallelograms are identically equal Q E D

COR *Two rectangles are equal, if two adjacent sides of the one are respectively equal to two adjacent sides of the other, and two squares are equal, if a side of the one is equal to a side of the other*

THEOREM 30

If a quadrilateral has two opposite sides equal and parallel, it is a parallelogram

Part. En. Let $ABCD$ be a quadrilateral in which the opposite sides AB, CD are equal and parallel,



it is required to prove that AC is equal and parallel to BD

Proof Join AD.

Then because AB is parallel to CD, (Hyp)
therefore the angle BAD is equal to the alternate angle ADC, (Th. 22)

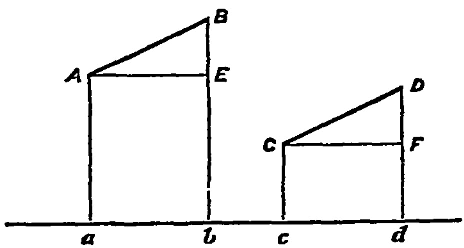
and therefore in the triangles BAD, CDA, we have

$$\left.\begin{array}{r} BA = CD, \qquad \text{(Hyp)} \\ AD \text{ common,} \\ \text{and the contained angles } BAD,\ CDA \\ \text{equal,} \end{array}\right\}$$

therefore the triangles are equal in all respects; (Th. 5.)
that is, $BD = AC$;
and the angle BDA = the angle CAD;
but these are alternate angles;
and therefore AC is parallel to BD*. (Th 21.) Q E D

§ THEOREM 31.

Straight lines which are equal and parallel have equal projections on any other straight line; conversely, parallel straight lines which have equal projections on another straight line are equal; and equal straight lines, which have equal projections on another straight line, are equally inclined to that line.



Part. En. Let AB, CD be equal and parallel straight lines, and let ab, cd be their projections on any other straight line. Then shall ab be equal to cd.

Proof. Through A, C draw AE, CF parallel to $abcd$, meeting Bb, Dd in E, F.

* Euclid, I. 33.

Then because BA, AE, BE are respectively parallel to DC, CF, DF, (Hyp.)

therefore the angle BAE = the angle DCF,

and the angle BEA = the angle DFC, (Th. 27. Cor 2.)

and the hypotenuse AB = the hypotenuse CD, (Hyp.)

therefore $AE = CF$. (Th. 17.)

but $AE = ab$, and $CF = cd$, (Th. 28.)

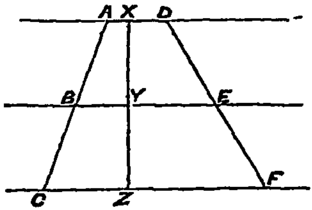
and therefore $ab = cd$. Q. E. D.

Similarly the converse propositions may be proved.

THEOREM 32.

If there are three parallel straight lines, and the intercepts made by them on any straight line that cuts them are equal, then the intercepts on any other straight line that cuts them are equal.

Let the three parallel straight lines AD, BE, CF make equal intercepts on the straight line AC, that is, let $AB = BC$.



Then shall the intercepts on any other line DEF be equal, that is, DE shall be equal to EF.

If one of the straight lines is perpendicular to the parallels, what is required to be proved follows directly from Theorem 31

But if neither of the straight lines is perpendicular to the parallels,
draw any straight line XYZ perpendicular to the parallels.

Then by Th 31, the equal straight lines AB, BC have equal projections;

and therefore $XY = YZ$.

And again by the same Theorem, because $XY = YZ$, and that these are the projections of DE, EF,

therefore $DE = EF$. Q E. D

Cor 1. *The straight line drawn through the middle point of one of the sides of a triangle parallel to the base passes through the middle point of the other side.*

Cor 2 *The straight line joining the middle points of two sides of a triangle is parallel to the base.*

Exercises for Solution.

1. If ABC is an isosceles triangle and A is double of either B or C, shew that A is a right angle.

2. If ABC is an isosceles triangle and A is half of either B or C, shew that A is two-fifths of a right angle.

3 Find the angle between the lines that bisect the angles at the base of the triangle in the last question.

4. The perpendiculars let fall from the extremities of the base of an isosceles triangle on the opposite sides will include an angle supplementary to the vertical angle of the triangle.

5. Shew that the angles of an equiangular triangle are equal to two-thirds of a right angle

6 Find the magnitude of the angle of a regular octagon (Th. 26)

7 How many equiangular triangles can be placed so as to have one common angular point, and fill up the space round it?

8. Shew that three regular hexagons can be placed so as to have a common point, and fill up the space round that point.

9 Shew that two regular octagons and one square have the same property.

Draw a pattern consisting of octagons and squares.

10 Shew that the angle of a regular pentagon is to the angle of a regular decagon as 3 to 4.

11 If a line is perpendicular to another it will be perpendicular to every line parallel to it

12. If a polygon is equilateral, does it follow that it is equiangular, and conversely?

13 How many diagonals can be drawn in a pentagon? How many in a decagon? How many in a polygon of n sides.

14 Shew that a square, a hexagon and a dodecagon will fill up the space round a point, and make a pattern of these polygons.

15. Examine whether a square, a pentagon and an icosagon have the same property; and also whether a pattern can be constructed of pentagons and decagons.

16. The exterior angle of a regular polygon is one-third of a right angle: find the number of sides in the polygon.

17. Two lines intersecting in A are respectively perpendicular to two lines intersecting in B: prove that any angle at A is equal or supplementary to any angle at B.

18. Shew that a trapezium may be divided into a parallelogram and a triangle.

19 The diagonals of any parallelogram will bisect one another.

20 The diagonals of a rhombus will bisect one another at right angles.

21. If two straight lines be drawn bisecting one another, and their extremities be joined, the figure so formed will be a parallelogram.

22. Given that a four-sided figure has its opposite sides equal, prove that it must be a parallelogram.

23 Prove that the diagonals of a rectangle are equal to one another.

24. Shew that if one element (a side) is given, a square is determined, if two elements (a side and angle), a rhombus is determined; also that if two elements (two sides) are given, a rectangle is determined: and find the number of elements required to determine a parallelogram, a trapezium, a quadrilateral, a pentagon, and a polygon of any number (n) of sides.

QUESTIONS ON SECTION III.

1. Give the derivation of the words parallel, parallelogram, trapezium.

2. What is indirectly ascertained in Theorem 21? Would it be possible to ascertain it directly?

3 Prove Th 24 by drawing a straight line to intersect *A*, *B* and *X*, and using Theorems 21, 22

4 Given two angles of a triangle to be respectively 72^0 $15'$ $47''$ and $83^0 . 51' . 16''$, find the third angle

5 If one angle of a triangle is equal to the other two, prove that it must be a right angle

6 Isosceles triangles having equal vertical angles must have equal base angles.

SECTION IV.

Problems.

In the Science of Geometry there are not only theorems to be proved, but constructions to be effected, which are called *problems*. Geometers have always imposed certain limitations on themselves with respect to the instruments which might be used in these constructions. There is no reason why any convenient instrument used in the Art of Geometry, such as the square, parallel ruler, sector, protractor, should not be supposed to be used also in the Science; but the ruler and compasses suffice for nearly all the simpler constructions, and those which cannot be effected by their means are considered as not forming a part of Elementary Geometry. These instruments are therefore *postulated* or requested (vid. p. 4). There are some problems, that seem at first sight not very difficult, that cannot be solved by the use of these instruments. We can, for example, bisect an angle; but we cannot, in general, trisect it, that is, divide it into three equal parts, by any combination of ruler and compasses.

It may be observed that the ruler is simply a straight edge, not graduated, and the compasses are supposed to be transferable from one part of the figure to another, the distance between the points being unaltered.

The solution of a problem in Elementary Geometry as above defined consists

(1) in indicating how the ruler and compasses are to be used in effecting the construction required,

(2) in proving that the construction so given is correct,

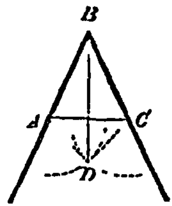
(3) in discussing the limitations, which sometimes exist, within which alone the solution is possible.

We shall give several examples of such problems, and then discuss the principles of the methods we have used.

PROBLEM 1

To bisect a given angle

Construction Let ABC be the given angle.



Take any equal lengths BA, BC, along its arms, and join AC

With centre A, and any radius greater than half AC, describe a circle, and with centre C, and the same radius, describe another circle intersecting the former circle on the side of AC remote from B in D.

Join AD, CD, and BD,
BD bisects the angle ABC.

Proof. In the triangles ABD, CBD,

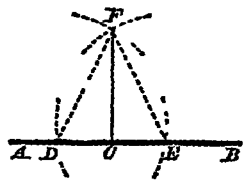
because $AB = BC$, (Constr)
and BD is common,
and the base AD = the base DC, (Constr)

therefore the angle ABD = the angle CBD, (Th 15)
that is, BD bisects the angle ABC*.

* Eucl 1 9

Problem 2.

To draw a perpendicular to a given straight line from a given point given in it.



Construction. With centre C and any radius describe a circle to cut the straight line in two points D, E, so that $CD = CE$.

With centre D, and any radius greater than DC, describe a circle, and with centre E and the same radius describe a circle, cutting the former in F.

Join FC;

it is required to prove that FC is perpendicular to AB.

Proof. In the triangles DCF, ECF,

because	$DC = CE$,	(Constr.)
CF is common,		
and the base $DF =$ the base EF,		(Constr.)
therefore the angle $DCF =$ the angle ECF,		(Th. 15.)
and therefore DCF and ECF are right angles*.		(Def. 14.)

Note.—This construction is usually effected in practice by means of the square.

It may be observed that this problem is only a special case of Prob. 1, the given angle being a straight angle.

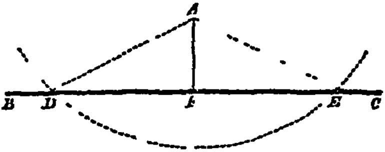
Problem 3.

To draw a perpendicular to a given straight line from a given point outside it.

Let BC be the given straight line, A the given point.

* Euclid, I. 11.

Construction With centre A describe a circle with any sufficient radius to cut BC in two points D, E.



Bisect the angle DAE by the line AF. (Prob 1)

Then AF shall be perpendicular to BC.

Proof. In the triangles AFD, AFE,

because $AD = AE$, (Constr)

and AF is common,

and the contained angle DAF = the contained angle EAF, (Constr)

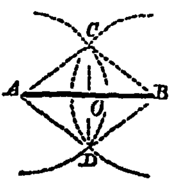
therefore the angle AFD = the angle AFE; (Th 5)

therefore AF is perpendicular to DE*.

NOTE—This construction also is usually effected in practice by means of the square

PROBLEM 4.

To bisect a given straight line†.



Let AB be the given straight line

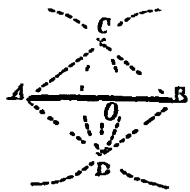
Construction With centre A and any radius greater than half AB describe a circle, and with centre B and the same radius describe a circle intersecting the former in two points C and D.

Join CD cutting AB in O

* Euclid, I. 12. † Euclid, I 10

Then O will be the point of bisection. Join AD, DB.

Proof. Because $AC = CB$; and CD is common to the
ACD two triangles $\underline{ACB}$, DCB; and the base AD is equal to the base DB; therefore the angle ACD = the angle BCD; therefore in the two triangles ACO, BCO,



we have $AC = BC$, (Constr.)
CO common,
and the included angles ACO, BCO equal;

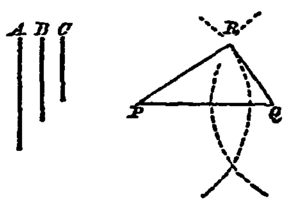
therefore the base AO = the base BO, (Th. 5)

or the line AB is bisected in O.

PROBLEM 5.

To construct a triangle, having given the lengths of the three sides.

Let the three given lengths be the lines A, B, C.



Construction. Draw a line PQ equal to one of them A. With centre P and radius equal to B describe a circle; and with centre Q and radius equal to C describe a circle. Let these circles intersect in R. Join RP, RQ.

RPQ is the triangle required.

Proof. For RPQ has its three sides respectively equal to A, B and C*.

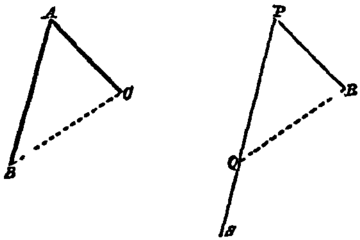
* Euclid, I. 22.

Limitation —It is necessary that any two of the lines A, B, C should be together greater than the third For if B and C were together less than A, the circles in the figure would obviously not meet. and if they were together equal to A, the point R would be on PQ, and the triangle would become a straight line Similarly if B were greater than $A + C$ or C greater than $A + B$, the circles would not intersect This limitation might be anticipated from the theorem before proved, that any two sides of a triangle are together greater than the third side, and is in fact its contrapositive.

23 PROBLEM 6.

At a given point in a given straight line to make an angle equal to a given angle.

Let BAC be the given angle, P the given point in the line PQ



Constr Join any two points B, C in the arms of the given angle Construct a triangle PQR having its three sides PQ, QR, RP respectively equal to AB, BC, CA (Prob 5)

Proof. In the triangles ABC, PQR,

because $AB = PQ$, (Constr)

$AC = PR$, (Constr)

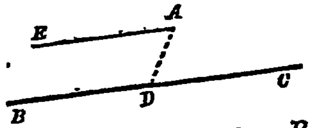
and $BC = QR$, (Constr)

therefore the angle $A =$ the angle P*. (Th 12)

* Euclid, I 23.

PROBLEM 7.

31 *To draw through any point a straight line parallel given straight line.*



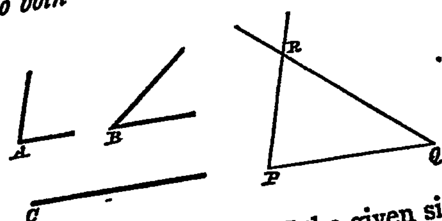
Constr. Let A be the given point, BC the given line. Draw any line AD to meet BC, and make the angle DAE equal to the alternate angle ADC. (Prob. 6.)

Proof. Because the alternate angles EAD, ADC are equal, (Constr.)

therefore AE is parallel to DE*. (Th. 21.)

PROBLEM 8.

To construct a triangle, having given two angles and a side adjacent to both



Let A, B be the two angles, C the given side.

Take a line $PQ = C$. At the points P, Q make an with PQ equal respectively to A and B. (Prob.

Let the lines which contain these angles meet in R.

Then RPQ is the triangle required.

Proof. For it has $PQ = C$, and the angles P a respectively equal to A and B.

* Euclid, I 31.

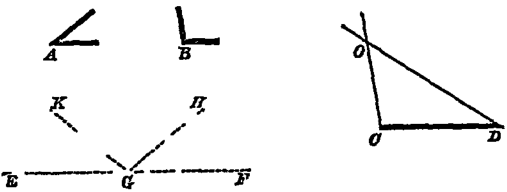
Limitation.—The two given angles must be together less than two right angles, or the lines PQ, QR would not meet. This follows also from the theorem that any two interior angles of a triangle are together less than two right angles, and is the contrapositive of that theorem.

PROBLEM 9

To construct a triangle, having given two angles and a side opposite to one of them

Let A and B be the given angles, CD the given side which is to be opposite to A.

Construction Draw an indefinite straight line EF. At any point G in it make the angles $FGH=A$, and $HGK=B$ (Prob 6), then KGE will equal the third angle of the triangle, since the sum of the three angles of



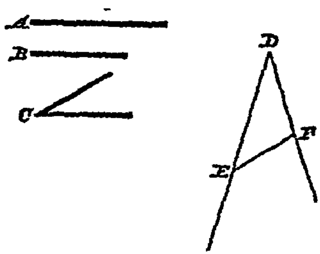
a triangle is equal to two right angles (Th 25) At C and D make angles equal to HGK and KGE, and let their sides meet in O, then OCD is the triangle required.

Proof. For OCD has CD equal to the given line, and the angles C and D equal respectively to the given angles

Limitation.—As before, the two given angles must be together less than two right angles

Problem 10.

To construct a triangle, having given two sides and the angle between them.



Let A, B be the given sides, C the given angle.

Construction. Draw an angle D equal to the given angle, and take DE, DF equal to A and B. Join EF.

Proof. For the triangle DEF has DE, DF equal to the given lines A and B, and the included angle D equal to the given angle C.

Remark In these problems we have found that one triangle and only one can be constructed to fulfil the conditions given. In other words, that with these *data* the triangle is *determinate.* Also we notice that in each case *three* elements in the triangle are *data* or given. We have given either the three sides, or two angles and the side adjacent to both, or two angles and a side opposite to one, or two sides and the included angle. And these cases correspond to the theorems proved above of the equality of triangles. For if *only one* triangle can be constructed so as to have its sides equal to three given lines, it is clear that if two triangles have the three sides of the one equal to the three sides of the other, these triangles must be identical, or be equal in all respects And a similar remark may be made on the other cases we have considered

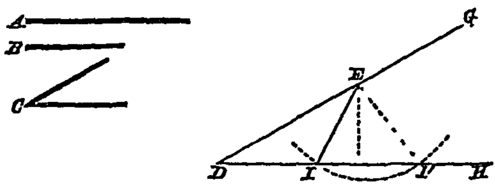
But there are cases in which the data may be insufficient to determine the triangle. For example, if only two sides are given, an

indefinite number of different triangles may be constructed to have these sides Or if the three angles are given, their sum being equal to two right angles, an indefinite number of triangles may be constructed to have these three angles And again it may be impossible to construct the triangle with the given data, as has been already shewn In some cases moreover the solution is *ambiguous, that is, there may be more than one triangle* which fulfils the given conditions. The following is an important instance of this, and is usually called *the ambiguous case*, some consideration of which occurred in Theorem 20

PROBLEM 11

To construct a triangle, having given two sides and an angle opposite to one of them.

Let A, B be the given sides, C the angle to be opposite to the side B.



Take an angle $GDH = C$, take $DE = A$, and with centre E and radius $= B$ describe a circle If I is one of the points in which this circle meets the line DH, by joining EI we obtain a triangle which fulfils the given conditions

But several cases may arise

Let the given angle be acute, as in the figure.

Then, by Theorem 19,

(1) If B is less than the perpendicular from E on DH, the circle would not meet DH, and the triangle would be *impossible.*

(2) If B is equal to the perpendicular, the circle would meet DH at the foot of the perpendicular, and there would be *one triangle, right-angled,* which fulfils the given conditions

(3) If B is greater than the perpendicular but less than DE, then the circle will meet DH in two points I, I' as in the figure, on the same side of D, and there will be *two triangles* EDI, EDI' which fulfil the given conditions.

(4) If B is equal to DE, the point I will coincide with D, and one of the two triangles disappears, and the other is isosceles.

(5) If B is greater than DE, the circle will meet DH in two points on the opposite sides of D, but one only of the triangles made by joining EI, EI' will be found to have the angle D, and the other will have the supplementary angle: that is, there will be only *one solution.*

The cases of the given angle being a right angle or an obtuse angle are left to the ingenuity of the student.

SECTION V

Loci

WHEN a point has to be found to fulfil one given geometrical condition, the problem is indeterminate that is, an infinite number of points can be found to fulfil the given condition For example, if the problem is to find a point at a given distance from a given point, it is plain that all the points in the circumference of a circle, described with that point as centre and the given distance as radius, fulfil this condition Or again, if a point has to be found at a given distance from a given straight line of indefinite length, it may lie anywhere on either of two straight lines parallel to the given line, and at the given distance from it on either side.

All the points which satisfy a single given geometrical condition lie in general in a line or lines · and this line, or these lines, are called the locus of the point under the given condition Hence we get the following definition of a locus .

Def. If any and every point on a line or group of lines (straight or curved), and no other point, satisfies an assigned condition, that line or group of lines is called the *locus* of the point satisfying that condition.

In order that a line or group of lines A may be properly termed the locus of a point satisfying an assigned condition X, it is necessary and sufficient to demonstrate the two following associated Theorems:

If a point is on A, it satisfies X.

If a point is not on A, it does not satisfy X.

It may sometimes be more convenient to demonstrate the contrapositive of either of these Theorems.

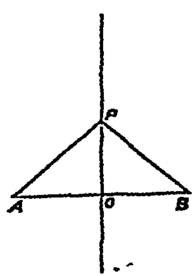
The following examples of loci are important.

i. *The locus of a point at a given distance from a given point is the circumference of a circle having a radius equal to the given distance and having its centre at the given point.*

ii. *The locus of a point at a given distance from a given straight line is the pair of straight lines parallel to the given line, at the given distance from it, and on opposite sides of it.*

The proofs of these two Theorems are obvious

iii. *The locus of a point equidistant from two given points is the straight line that bisects, at right angles, the line joining the given points.*



Part. En. Let A, B be the two given points; P a point equidistant from A and B, so that $PA = PB$;

it is required to find the locus of P.

Constr. Join AB; and bisect it in O, and join PO.

Then PO produced is the locus required.

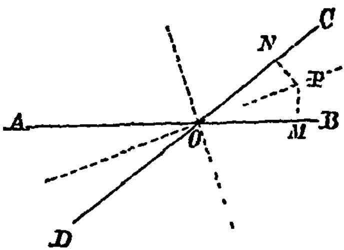
Proof In the triangles AOP, BOP,

because $AO = OB$, (Constr)
and PO is common,
and $AP = BP$, (Hyp)

therefore the angle $AOP =$ the angle BOP, (I. 15)
and therefore PO is at right angles to AB,
that is, a point equidistant from A and B lies on the line which bisects AB at right angles Further, every point not on the bisector, is at unequal distances from A and B, as may be proved by Theorem 14, and therefore the line which bisects AB at right angles is the locus of points equidistant from A and B

IV *The locus of a point equidistant from two intersecting straight lines is the pair of lines, at right angles to one another, which bisect the angles made by the given lines*

Let AB, DC intersect in O, it is required to find a point equally distant from AB and DC



Bisect the angle COB, and in the bisector take any point P. Let fall PN, PM perpendicular to DC, AB

In the triangles PON, POM,
because the angles PON, PNO are respectively equal to the angles POM, PMO, (Constr)

and the hypotenuse PO is common,
therefore the triangles are equal in all respects,
(Theorem 17)
and therefore $PN = PM$.

In the same manner every point in the bisector of any one of the four angles at O is equally distant from AB and CD;
that is, the locus of points equally distant from two straight lines which intersect, is the bisectors of the angles between the lines.

It may further be proved that no point not in a bisector is equally distant from these lines, that is, the bisectors are the *complete locus.*

Exercises.

Find the following loci.—

(1) Of a point at a given distance from a given point

(2) Of a point at a given distance from a given line.

(3) Of a point at a given distance from a given circle.

(4) A horse is tethered by a chain fastened to a ring which slides on a rod bent into the form of a rectangle Find the outline of the area over which he can graze.

(5) Find the locus of a point equidistant from two given points. Prove that the locus found is *complete.*

(6) Find the locus of points at which two equal lengths, adjacent or not adjacent, of a straight line subtend equal angles.

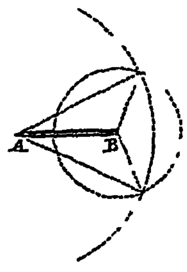
(7) Find part of the locus of points at which two adjacent sides of a square subtend equal angles

(8) Find the locus of a point at which two adjacent sides of a rectangle subtend supplementary angles

INTERSECTION OF LOCI

When a point has to be found which satisfies *two* conditions, the problem is generally determinate if it is possible and the method of loci is very frequently employed in discovering the point. For if the locus of points which satisfy each condition separately is constructed, it is obvious that the points which satisfy both conditions must be the points common to both loci, that is, must be the point or points where the loci intersect

For example, a triangle is to be constructed on a given base with its sides of given lengths Let AB be the base.



The two conditions are that the lengths of the two sides are given, the point sought for is the vertex · now the vertex must be at a certain distance from A = one of the given lengths, its locus is therefore a certain circle round A as centre Similarly it must be at a certain distance from B, its locus is therefore another circle round B as centre The points of intersection of these circles are therefore the vertices of the two equal triangles which fulfil the given conditions.

It was this reasoning that suggested the construction in Problem 5.

Occasionally it will be found that with certain conditions among the data in the following Exercises the loci do not intersect, or the solution becomes impossible. So in the case given, it will not be difficult to see that the circles would not intersect unless any two of the given sides were greater than the third side These conditions among the data for the possibility or impossibility of a solution should always be found.

The principle of the intersection of loci may be thus stated.

If A is the locus of a point satisfying the condition X, and B the locus of a point satisfying the condition Y; then the intersections of A and B, and these points only, satisfy both the conditions X and Y.

The following examples of intersection of loci are important, and are at once demonstrated by the aid of the preceding examples of loci.

i. *There is one and only one point in a plane which is equidistant from three given points not in the same straight line.*

ii. *There are four and only four points in a plane each of which is equidistant from three given straight lines that intersect one another but not in the same point.*

EXERCISES ON INTERSECTION OF LOCI.

1. Find a point in a given straight line at equal distances from two given points. Construct the figures for all cases.

2. Find a point in a given straight line at a given distance from a given straight line.

3 Find a point in a given straight line at equal distances from two other straight lines

4 On a given straight line to describe an equilateral triangle

5. Describe an isosceles triangle on a given base, each of whose sides shall be double of the base

6 Find a point at a given distance from a given point, and at the same distance from a given straight line

7. Given base, sum of sides, and one of the angles at the base, construct the triangle

8. Given base, difference of sides, and one of the angles at the base, construct the triangle

9 Find a point at a given distance from the circumference of two given circles, the distances being measured along their radii or their radii produced

10. A straight railway passes within a mile of a town A place is described as four miles from the town, and half a mile from the railway. How many places satisfy the conditions?

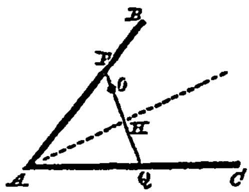
11. Find a point equidistant from three given straight lines that intersect one another but not in the same point

ANALYSIS AND SYNTHESIS

If problems cannot be solved by this method, it remains to attack them by the method, as it is called, of Analysis and Synthesis This is not so much a method as a way of searching for a suggestion, and nothing but experience and ingenuity will here avail the student. The solution is

supposed to be effected, and relations among the parts of the figure are then traced until some relation is discovered which can give a clue to the construction. Nothing but seeing examples can make this clear.

(1) *It is required to draw a line to pass through a given point and make equal angles with two given intersecting lines.*



Let O be the given point, AB, AC the given lines.

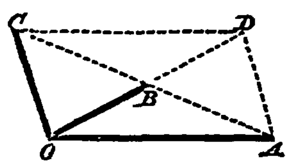
We reason as follows (*analysis*): suppose POQ were the line required, then the angle at P = angle at Q.

Therefore $AP = AQ$, therefore if we bisected the angle A, POQ would be at right angles to the bisector.

Now this is a suggestion we can work backwards from, and the construction is as follows.

Synthesis. Bisect the angle BAC, and let fall OH a perpendicular to the bisector, and let it meet the lines in P, Q, and POQ can then be proved to be the line required.

(2) *It is required to draw from a given point three straight lines of given lengths, so that their extremities may be in the same straight line, and intercept equal distances on that line.*



Analysis. Suppose OA, OB, OC were the three lines, so that CBA is a straight line, and $CB = BA$.

Then it occurs to us that if

OB were prolonged to D, making $BD = OB$, then CD and DA would be respectively parallel and equal to OA and OC (see § 3, Ex. 21), and that the sides of the triangle DOA are respectively equal to OA, OC and $2OB$. Hence the construction is suggested

Synthesis Make a triangle DOA whose sides are OA, OC, and $2OB$, complete the figure, by drawing DC, OC parallel to OA, AD; and the other diagonal ABC will be the line required For it may be shewn that $AB = BC$

The student must not be surprised if he finds problems of this class difficult For there is nothing except previous knowledge of geometrical facts to point out which of the many relations of the parts of the figure are to be followed up in order to arrive at the particular relation which suggests the construction It is not easy to see what is to suggest the producing of OB to D as in the figure.

Subjoined are a few problems of no great difficulty, which may be solved by this method

PROBLEMS.

1 On a given straight line to describe a square.

2. Describe a rectangle with given sides

3 Given two sides of a parallelogram and the included angle, construct the parallelogram

4 Given the lengths of the two diagonals of a rhombus, construct it

5. From a given point without a given straight line to draw a line making an angle with the line equal to a given angle

6. Describe a square on a given straight line as diagonal.

7. Draw through a given point, between two straight lines not parallel, a straight line which shall be bisected in that point.

8. Place a line of given length between two intersecting lines so as to be parallel to another given line.

9. Trisect a right angle.

10. Divide half a right angle into six equal parts

11. Three straight lines meet in a point, draw a straight line such that the parts of it intercepted by the three lines shall be equal to one another.

12. Trisect a given straight line.

BOOK II.*

EQUALITY OF AREAS.

SECTION I

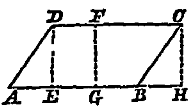
THEOREMS

IN Book I Theorems 5, 7, 15, 17, 20 we have had instances of figures whose areas are equal, and whose areas are proved to be equal, by shewing that the figures could be placed so as to coincide with one another, or are *congruent*, or identically equal But figures of different shapes may nevertheless be equal in area, though they cannot be placed so as to coincide with one another, thus a circular field may be as large as a square one, and a triangle as large as a rectangle

In the present section we proceed to the consideration of rectilineal figures whose areas are equal, though the figures are not of the same shape.

Def 1 The *altitude* of a parallelogram with reference to a given side as base is the perpendicular distance between the base and the opposite side

Thus in the figure the perpendiculars *DE*, *FG*, or *CH*, which are equal (by I 28) since *DEFG*, *DEHC* are parallelograms, are each of them the altitude of the parallelogram *ABCD*, *AB* being the base



* Book III (with the exception of its last Section) is independent of Book II., and may be studied immediately after Book I

Def. 2. The *altitude* of a triangle with reference to a given side as base is the perpendicular distance between the base and the opposite vertex.

Obs. It follows from the General Axioms (*d*) and (*e*) (page 1), as an extension of the Geometrical Axiom 1 (page 11), that magnitudes which are either the sum or the difference of identically equal magnitudes are equal, although they may not be identically equal.

THEOREM 1.

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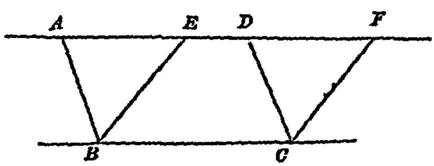
Parallelograms on the same base and between the same parallels are equal.

Part En. Let $ABCD$, $EBCF$ be parallelograms, upon the same base BC, and between the same parallels AF, BC;



it is required to prove that the parallelogram $ABCD$ is equal to the parallelogram $EBCF$.

Proof. Because $ABCD$ is a parallelogram, (Hyp.)
therefore $AB = DC$; (I. 28.)
because AB, BE are respectively parallel to CD, CF, (Hyp.)
therefore the angles at A and E are respectively equal to the corresponding angles at D and F; (I. 23, Cor.)
therefore the triangles ABE, DCF are equal. (I. 17.)

But if the triangle *CDF* is taken away from the trapezium *ABCF* the parallelogram *ABCD* remains, and if the triangle *ABE* is taken away from the same trapezium the parallelogram *EBCF* remains, therefore the parallelogram *ABCD* is equal to the parallelogram *EBCF**

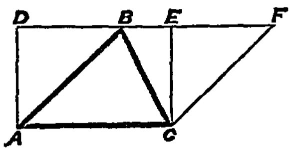
COR 1 *The area of a parallelogram is equal to the area of a rectangle, whose base and altitude are equal to those of the parallelogram*

COR 2 *Parallelograms on equal bases and of equal altitude are equal*†, *and of parallelograms of equal altitudes, that is the greater which has the greater base, and also, of parallelograms on equal bases, that is the greater which has the greater altitude*

THEOREM 2.

The area of a triangle is half the area of a rectangle whose base and altitude are equal to those of the triangle

Let *ABC* be a triangle on the base *AC*, and *DACE* the rectangle on the same base, and having the same altitude as the triangle,



then will the area of the triangle *ABC* be half that of the rectangle *DACE*.

* Euclid, I 35 † Euclid, I 36

Constr Through C draw CF parallel to AB, to meet DE produced in F.

Then $BACF$ is a parallelogram, and therefore $BACF$ is equal to the rectangle $DACE$. (II. 1.)

But the triangle ABC is half the parallelogram $BACF$; (I. 28.)

therefore the triangle ABC is half the rectangle $DACE$

COR. 1. *Triangles on the same or equal bases and of equal altitude are equal*.*

COR 2. *Equal triangles on the same or equal bases have equal altitudes.*

COR. 3. *If two equal triangles stand on the same base and on the same side of it, or on equal bases in the same straight line and on the same side of that straight line, the line joining their vertices is parallel to the base or to that straight line†.*

THEOREM 3.

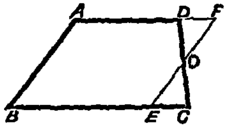
The area of a trapezium is equal to the area of a rectangle whose base is half the sum of the two parallel sides, and whose altitude is the perpendicular distance between them

Part En. Let $ABCD$ be a trapezium having AD parallel to BC;

then it is required to prove that its area is equal to

* Euclid, I. 37, 38. † Euclid, I 39, 40.

that of the rectangle whose base is half the sum of AD and BC, and altitude the perpendicular distance between AD and BC.



Proof Bisect DC in O, and through O draw a line parallel to AB to meet BC in E, and AD produced in F.

Then in the triangles DOF, EOC,

because $DO = OC$, (Constr.)
and the angle DOF = the angle EOC, (I. 4.)
and the angle ODF = the angle OCE, (I. 22.)

therefore the triangles are equal in all respects; (I. 7.)

and therefore the trapezium $ABCD$ is equal to the parallelogram $ABEF$.

But the parallelogram $ABEF$ is equal to the rectangle on the same base BE, and between the same parallels; (II. 1. Cor.)

and since $EC = DF$,

and $AF = BE$,

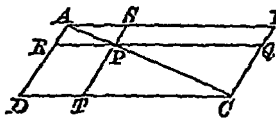
therefore the base BE is half the sum of AD and BC;

therefore the trapezium $ABCD$ is equal to the rectangle whose base is half the sum of the parallel sides, and height the perpendicular distance between them.

Def. 3. The straight lines drawn through any point in a diagonal of a parallelogram parallel to the sides divide it into four parallelograms, of which the two whose diagonals are upon the given diagonal are called *parallelograms about that diagonal*, and the other two are called the *complements* of the parallelograms about the diagonal.

THEOREM 4.

The complements of parallelograms about the diagonal of any parallelogram are equal to one another.



Let $ABCD$ be a parallelogram, P any point on the diagonal AC, and let RPQ, SPT be drawn parallel to the sides,

it is required to prove that the complement PB = the complement PD.

Proof. For the triangle ABC = the triangle ADC (I 28); and the triangles ASP, PQC = the triangles ARP, PTC, therefore the remainders are equal, that is, $PB = PD$*.

Def. 4. All rectangles being identically equal which have two adjoining sides equal to two given straight lines, any such rectangle is spoken of as *the rectangle contained by those lines*

In like manner, any square whose side is equal to a given straight line is spoken of as *the square on that line.*

Def 5. A point in a straight line is said to divide it *internally*, or, simply, to divide it, and, by analogy, a point

* Euclid, I 43.

in the line produced is said to divide it *externally*, and, in either case, the distances of the point from the extremities of the line are called its *segments*

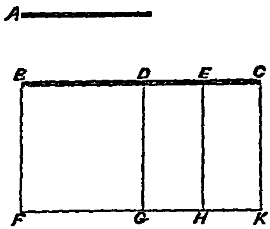
Obs A straight line is equal to the sum or difference of its segments according as it is divided internally or externally.

THEOREM 5

The rectangle contained by two given lines is equal to the sum of the rectangles contained by one of them and the several parts into which the other is divided

Part En Let A and BC be the two given lines, of which BC is divided into any number of parts, BD, DE, EC,

it is required to prove that the rectangle contained by A and BC is equal to the sum of the rectangles contained by A and BD, A and DE, A and EC.



Proof. From B draw a line BF at right angles to BC, and equal to A, through F draw a line parallel to BC, and through D, E, C draw DG, EH, CK parallel to BF

Then the figure BK is equal to the figures BG, DH, EK:

but BK is the rectangle contained by A and BC,

and BG, DH, EK are respectively the rectangles contained by A and BD, A and DE, A and EC,

therefore the rectangle contained by A and BC is equal to the rectangles contained by A and BD, A and DE, A and EC*.

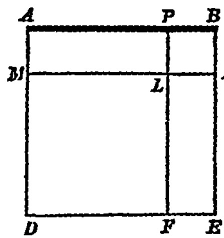
COR 1. *If a straight line is divided into two parts, the rectangle contained by the whole line and one of the parts is equal to the sum of the square on that part and the rectangle contained by the two parts*†.

COR 2. *If a straight line is divided into two parts the square on the whole line is equal to the sum of the rectangles contained by the whole line and each of the parts*‡.

THEOREM 6.

The square on the sum of two lines is greater than the sum of the squares on those lines by twice the rectangle contained by them§

Part En. Let AB be the sum of AP, PB;



it is required to prove that the square on AB is equal to the squares on AP, PB together with twice the rectangle contained by AP, PB.

Proof. Describe a square $ADEB$ on AB.

Through P draw PLF parallel to AD, meeting DE in F: cut off $PL=PB$ leaving $LF=AP$. Through L draw HLM parallel to AB, to meet DA and EB in M, H.

* Euclid, II 1. † Euclid, II 3 ‡ Euclid, II 2. § Euclid, II. 4

Then the figures AL, PH, LE, MF are rectangles by construction,

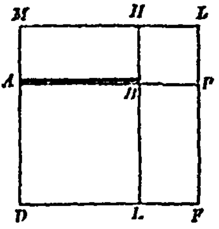
and PH, MF are the squares on PB, AP respectively, and AL, LE are each of them the rectangle contained by AP, PB.

Hence, since $ADEB$ is made up of these four figures, it follows that the square on AB is greater than the squares on AP, PB by twice the rectangle contained by AP, PB

THEOREM 7.

*The square on the difference of two lines is less than the sum of the squares on those lines by twice the rectangle contained by them**.

Part En Let AB be the difference of AP, BP, it is required to prove that the square on AB is less than the squares on AP, PB by twice the rectangle AP, PB.



Proof Describe a square $ADEB$ on AB

Through P draw LPF parallel to AD, meeting DE produced in F. cut off $PL = PB$, making $LF = AP$ Through L draw MHL parallel to AB, to meet DA and EB produced in M, H.

Then MF is the square on AP; and HP the square on BP, and MP or HF, the rectangle contained by AP and BP.

* Euclid, II 7

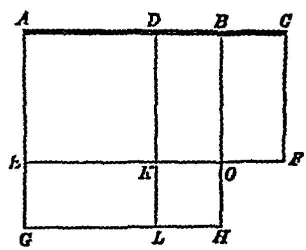
And AE is less than $MF + HP$ by $MP + HF$; that is, the square on AB is equal to the squares of AP, PB diminished by twice the rectangle contained by AP, PB.

THEOREM 8.

The difference of the squares on two lines is equal to the rectangle contained by the sum and difference of the lines.*

Part. En. Let AB and BC be the two straight lines, of which AB is the greater; and let them be placed in one straight line; cut off BD equal to BC; so that AC is their sum, and AD is their difference;

Then will the difference of the squares of AB and BC be equal to the rectangle contained by AC and AD.



Proof. On AB describe a square $AGHB$. Through D, C draw DL, CF parallel to AG or BH, cut off $HO = LH$ or DB; and through O draw $EKOF$ parallel to AC.

Then KH is the square on DB or BC, and therefore the difference of the squares of AB and BC is the figure made up of EL and AO.

But EL is equal to BF by construction;
therefore the figure made of EL and AO is equal to AF, which is the rectangle contained by AE or AD and AC;

* Euclid, II. 5, Cor.

therefore the difference of the squares of AB and BC is equal to the rectangle contained by AC and AD.

COR. *If a straight line is bisected and divided in any point, the rectangle contained by the segments is equal to the difference of the squares on half the line and the line between the points of section*

Proof For let AB be bisected in C, and divided internally or externally in P.

Then AP is the sum of AC and CP, and PB is their difference, since $BC = AC$.

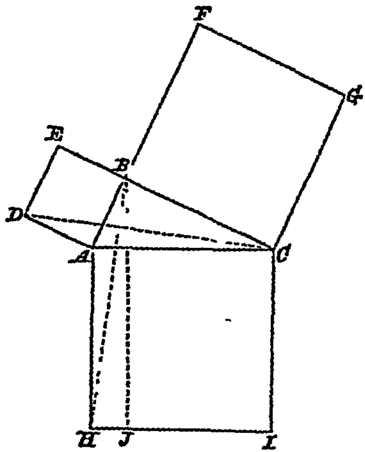
Therefore the rectangle contained by AP, PB is the rectangle contained by the sum and difference of AC and CP, and therefore is equal to the difference of the squares of AC and CP.

Remark The student will begin here to suspect, what he will afterwards find to be true, that there is an intimate relation between geometry and algebra. Algebraical or analytical geometry as it is called, investigates this relation and applies it to the establishment of theorems in geometry, and will occupy him at a later stage of his mathematical studies. We shall at present use the expression AB^2, which is read 'AB squared,' only as an *abbreviation* for "the square on AB," and $AB \times AC$ or $AB \cdot AC$, as an abbreviation for "the rectangle contained by AB and AC."

THEOREM 9

In any right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the sides which contain the right angle

Part. En. Let ABC be a triangle right-angled at B; it is required to prove that AC^2 is equal to AB^2+BC^2.



Proof. On AB, BC, CA describe the squares $ADEB$, $BFGC$, $CIHA$ respectively. Join CD, BH; and draw BJ parallel to AH.

Since the angles ABC, ABE, BCF are right angles, it follows that CBE, ABF are straight lines; (I. 3) therefore the triangle DAC is on the same base DA, and between the same parallels DA, EC with the square $DABE$;

therefore the triangle DAC is half the square $DABE$; (II. 3, Cor 2.)

and similarly the triangle BAH is half the rectangle AJ.

But because the angle DAB = the angle HAC, each being a right angle; add to each the angle BAC;

therefore the whole angle DAC is equal to the whole angle BAH;

and the two sides DA, AC are respectively equal to the two sides BA, AH; (Constr.)

therefore the triangle DAC is equal to BAH; (I. 5)

and therefore the square $DABE$ = the rectangle AJ.

Similarly it may be shewn that the square $BCGF$ = the rectangle CJ, and therefore, since AJ and CJ make up

the whole square *AHIC*, the square *AHIC* is equal to the sum of the squares *ABDE* and *BCGF*, that is,

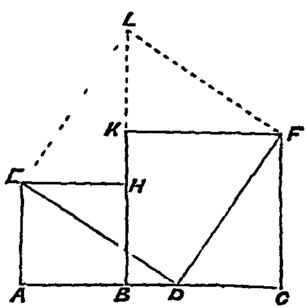
$$AC^2 = AB^2 + BC^2 \text{*}$$

This important proposition may be proved as follows

Place two squares *EABH*, *KBCF*, as in the figure, with their sides *AB*, *BC* continuous and in the same straight line

From *CB* cut off *CD* equal to *AB* Join *DE*, *DF*

Produce *BK* to *L*, making $KL = AB$. Join *LE*, *LF*



Then it will be easy to prove that the triangles *EAD*, *DCF*, *EHL*, *LKF* are all equal, being right-angled, and having the sides containing the right angle equal, therefore the figure *LEDF* is equal to the sum of the two given squares, and all its sides are equal

And since *EDA* is complementary to *AED* or *FDC*, therefore the angle *EDF* is a right angle

Therefore *LEDF* is a square, and is the square on *ED*

Therefore the square on the hypotenuse *ED* is equal to the sum of the squares on the sides *EA*, *AD*.

* Euclid, I 47

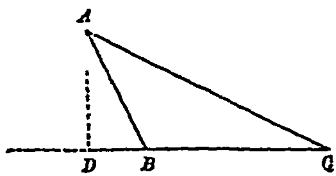
COR. 1. *It follows that in a triangle ABC right-angled at B,*

$$AB^2 = AC^2 - BC^2 \text{ and } BC^2 = AC^2 - AB^2.$$

The next two theorems shew the modifications which the theorem undergoes when the triangle is not right-angled.

THEOREM 10

In an obtuse-angled triangle the square on the side subtending the obtuse angle is greater than the squares on the sides containing that angle by twice the rectangle contained by either of these sides and the projection on it of the other side.*



Part En. Let ABC be the triangle, ABC being the obtuse angle, BD the projection of AB on BC, BC being produced backward.

Then will $AC^2 = AB^2 + BC^2 + 2CB \cdot BD$,

for $AC^2 = AD^2 + DC^2$, by (II. 9,)

but $AD^2 = AB^2 - BD^2$,

and $DC^2 = CB^2 + BD^2 + 2CB \cdot BD$, (by II. 6,)

therefore $AC^2 = AB^2 + BC^2 + 2CB \cdot BD$.

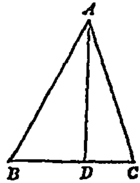
* Euclid, II. 12.

THEOREM 11.

In any triangle the square on the side opposite an acute angle is less than the squares on the other two sides by twice the rectangle contained by either side and the projection on it of the other side.*

Part. En Let ABC be a triangle, B an acute angle, BD the projection of AB on BC, then will

$$AC^2 = AB^2 + BC^2 - 2CB \times BD$$



Proof. For $AC^2 = AD^2 + DC^2$, (by II 9),

but $AD^2 = AB^2 - BD^2$, by the same Theorem,

and $DC^2 = BC^2 + BD^2 - 2CB \times BD$, (by II 7),

therefore $AC^2 = AB^2 + BC^2 - 2CB \times BD$

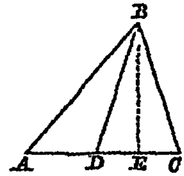
COR. *Conversely, the angle opposite a side of a triangle is an acute angle, a right angle, or an obtuse angle, according as the square on that side is less than, equal to, or greater than the sum of the squares on the other two sides.*

THEOREM 12

The sum of the squares on two sides of a triangle is double the sum of the squares on half the base and on the line joining the vertex to the middle point of the base.

* Euclid, II 13

Part. En Let AC, a side of the triangle ABC, be bisected in D, then will



$$AB^2 + BC^2 = 2AD^2 + 2BD^2.$$

Proof. For let DE be the projection of BD on AC.

Then $AB^2 = AD^2 + DB^2 + 2AD . DE$ (by II. 10),

and $\quad BC^2 = CD^2 + DB^2 - 2CD . ED$ (by II. 11),

therefore remembering that $AD = DC$, we obtain by addition that

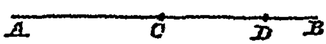
$$AB^2 + BC^2 = 2AD^2 + 2DB^2.$$

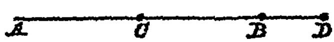
This theorem in a more general form is known as Apollonius's Theorem.

THEOREM 13.

If a straight line is divided internally or externally at any point, the sum of the squares on the segments is double the sum of the squares on half the line and on the line between the point of division and the middle point of the line.*

Let AB be bisected in C, and divided internally or externally in D.





Then the squares on AD, DB will be double of the squares on AC, CD.

Proof. For $AD^2 = AC^2 + CD^2 + 2AC \times CD$ by II 6;

and $DB^2 = CB^2 + CD^2 - 2BC \times CD$ by II. 7;

therefore, adding, and remembering that $AC = BC$, and that therefore the rectangle $AC \times CD$ = the rectangle $BC \times CD$, we get that $AD^2 + DB^2 = 2AC^2 + 2CD^2$.

* Eucl II 9, 10

EXERCISES

1 Bisect a triangle by a line passing through one of its angular points

2 Any line drawn through the intersection of the diagonals of a parallelogram to meet the sides bisects the figure

3 Find the locus of the vertices of triangles of equal area upon the same base

4 If the sides of a triangle are 3, 4, 5 inches respectively, the triangle is right-angled

5 Of all triangles having the same vertical angle, and whose bases pass through a given point, the least is that whose base is bisected in that point

6 The diagonals of a parallelogram divide it into four equivalent triangles

7. If from any point in the diagonal of a parallelogram straight lines be drawn to the angles, then the parallelogram will be divided into two pairs of equivalent triangles

8 $ABCD$ is a parallelogram, and E any point in the diagonal AC produced Shew that the triangles EBC, EDC will be equivalent

9. $ABCD$ is a parallelogram, and O any point within it, shew that the triangles OAB, OCD are together equivalent to half the parallelogram

10 On the same supposition if lines are drawn through O parallel to the sides of the parallelogram, then the difference of the parallelograms DO, BO is double of the triangle OAC.

11. The diagonals of a parallelogram $ABCD$ intersect in O, and P is a point within the triangle OAB. Prove that the difference of the triangles APB, CPD, is equivalent to the sum of the triangles APC, BPD.

12. If the points of bisection of the sides of a triangle be joined, the triangle so formed shall be one-fourth of the given triangle.

13. Shew that the sum of the squares on the lines joining the angular points of a square to any point within it is double of the sum of the squares on the perpendiculars from that point on the sides.

14. If the sides of a quadrilateral figure be bisected, and the points of bisection joined, prove that the figure so formed will be a parallelogram equal in area to half the given quadrilateral.

15. Bisect a parallelogram by a line passing through any given point.

SECTION II.

Problems.

On the Quadrature of a Rectilineal Area.

There is one problem which from its historical interest, and from the valuable illustrations it affords of the methods and limitations of Geometry, should find a place there. This problem is called *the quadrature of a rectilineal area*, which means the finding a square whose area is equivalent to that of any given figure which is bounded by straight lines. It gave a means of comparing any two dissimilarly

shaped rectilineal figures, such as irregularly shaped fields whose boundaries were straight In the present condition of mathematics it is not necessary, as the student will hereafter learn, but it will always be instructive.

The problem is approached by the following stages

(1) To construct a parallelogram, with sides inclined at a given angle, equal to a given triangle

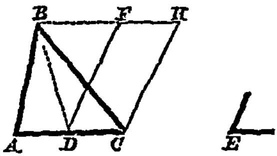
(2) To construct *on a given straight line* a parallelogram, with sides inclined at a given angle, equal to a given triangle

(3) To construct a parallelogram, with sides inclined at a given angle, equal to a *given rectilineal figure*

(4) To construct a *square* equal to a given rectilineal figure

PROBLEM I

To construct a parallelogram equal to a given triangle and having one of its angles equal to a given angle



Let ABC be the given triangle, E the given angle

Construction Bisect AC in D, make the angle $CDF=E$, and through B draw BFH parallel to AC, and draw CH parallel to DF

$FDCH$ will be the parallelogram required

Proof If BD be joined, it will be clear that the triangle BAC and the parallelogram $FHCD$ are each of them double of the triangle BDC (II 2, Cor. 1), and therefore the parallelogram $FHCD$ = the triangle BAC, and it has an angle $=E$, which was required*.

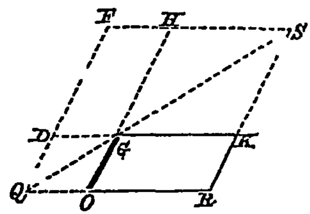
* Euclid, I 42.

Problem II.

To construct a parallelogram on a given base equal to a given triangle and having one of its angles equal to a given angle

Let BAC be the given triangle, E the given angle as before, and let it be required to construct on the line GO a parallelogram equal to BAC, and having an angle E

Construction Construct the parallelogram $FDGH$ as before, and place it so that one of its sides GH may be in the same straight line with GO



Produce FD, and draw OQ parallel to GD to meet FD in Q. Join QG, and produce it to meet FH produced in S

Draw SKR parallel to FQ, meeting DG produced in K, and QO produced in R.

Then $GORK$ is the parallelogram required

Proof For the parallelogram FG = the parallelogram GR, being complements (II 4), and FG = the given triangle ABC

Therefore GR = the triangle ABC, and it has an angle = E, since it is equiangular with the parallelogram $FDGH$*.

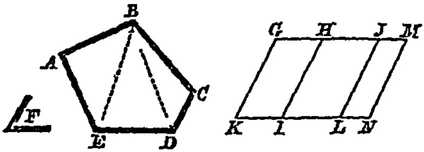
Problem III

To construct a parallelogram equal to a given rectilineal figure and having one of its angles equal to a given angle.

* Euclid, I 44.

Let *ABCDE* be the given rectilineal figure, *F* the given angle Divide *ABCDE* into triangles by joining *BE*, *BD*.

Construction Construct as before a parallelogram *GHIK* = *BAE*, and having an angle at *K* = *F*



Construct on *HI* a parallelogram *HJLI* = *BED*, and having the angle *HIL* = *F*

And construct on *JL* a parallelogram *JMNL* = *BCD* and having the angle *JLN* = *F*

GKNM will then be the parallelogram required

Proof. For since the angle *HIL* = the angle *K*, it is therefore supplementary to *HIK*, and therefore (by I 3) *KIL* is a straight line

Similarly *GM* and *KN* are straight lines, and *MN* is parallel to *GK* (I 24)

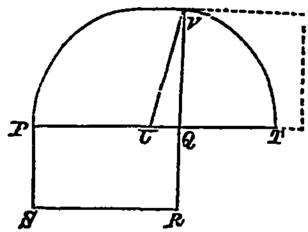
Therefore *GKNM* is a parallelogram, having the given angle, and it is by construction equal to the given rectilineal figure*

PROBLEM IV

To construct a square equal to a given rectilineal figure

* Euclid, I 45

Construction. By the previous construction make a rectangle equal to $ABCDE$, and let $PQRS$ be the rectangle so made



Then if $PQ = QR$ the rectangle is a square, but if not, produce PQ to T, making $QT = QR$; on PQT as diameter describe a semicircle, U being the centre, and produce RQ to meet the circumference in V.

If a square be described on QV, this square will be equal to $ABCDE$

Proof. For since PQ is the sum of PU and UQ, and QT is the difference of PU (or UT) and UQ, it follows (from II 8) that the rectangle $PQ \times QT = PU^2 - UQ^2$, but $PU^2 = UV^2$, and therefore $PU^2 - UQ^2 = UV^2 - UQ^2$, that is, VQ^2, by II 9, Cor.

But the rectangle $PQ \times QT$ is the rectangle $PQRS$, which was made equal to $ABCDE$.

Therefore $VQ^2 = ABCDE$, and the square described on VQ is the square required*.

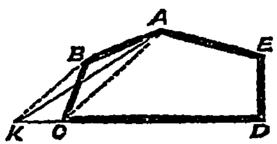
Remark If the given figure is not rectilineal, it cannot be divided into triangles· hence it is impossible by this method to construct a square equal to a given curvilinear area Nor can any method depending on the use of the ruler and compasses only (see p 60), construct a square equal to some curvilinear areas, such as the circle This is the problem of squaring the circle, the solution of which cannot be effected without the use of other instruments.

* Euclid, II 14.

PROBLEM V.

To construct a rectilineal figure equal to a given rectilineal figure and having the number of its sides one less than that of the given figure, and thence to construct a triangle equal to a given rectilineal figure

Let $ABCDE$ be the given rectilineal figure Join AC, and through B draw BK parallel to AC to meet DC produced in K. Join AK



Then since the triangle ABC is equal to the triangle AKC, being on the same base AC and between the same parallels, add to each $ACDE$, therefore the figure $ABCDE$ is equal to the figure $AKDE$, which has the number of its sides diminished by one

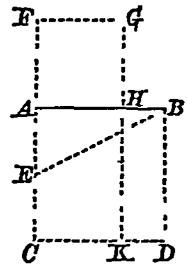
Since this process can be repeated any number of times it is evident that any polygon can be reduced in this manner to an equivalent triangle.

PROBLEM VI.

To divide a straight line, either internally or externally, into two segments such that the rectangle contained by the given line and one of the segments may be equal to the square on the other segment.

Let AB be the given line.

First, to divide it internally.



Construction Draw a square $ACDB$ on AB, bisect AC in E. Join BE, produce EA to F, making $EF=EB$, on AF describe a square $AFGH$.

AH and HB are the parts required, so that the rectangle $AB \times BH = AH^2$.

Proof. Produce GH to meet CD in K.

Then since CA is bisected in E, and divided externally in F,

therefore $CF \times FA = EF^2 - EA^2$ (II. 8 Cor.),

but $EF^2 = EB^2$, and therefore $EF^2 - EA^2 = AB^2$ (II 9 Cor),

therefore $CF \times FA = AB^2$;

that is, the figure FK = the figure AD, take from each AK, and therefore $FH = HD$.

But HD is the rectangle $AB \times BH$, and FH is the square on AH.

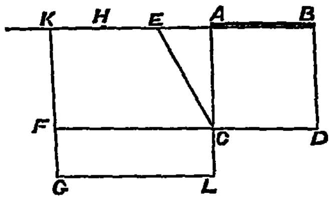
therefore $AB \times BH = AH^2$ *

Cor *The line* FA *is* divided externally *in* C, *so that*

$$FC . FA = CA^2.$$

* Euclid, II. 11.

Secondly, to divide it externally



Construction Produce BA, and take AH equal to AB Bisect AH in E On AB describe a square $ACDB$. Join EC, from EH produced, cut off $EK=EC$

Then will $AB \quad BK=AK^2$

Proof On KA describe a square $KGLA$, and produce DC to meet KG in F

Then, since AH is bisected in E and produced to K,

$$AK \times KH = EK^2 - EA^2, \qquad \text{(II 8)}$$

but $EK^2 = EC^2$, and therefore $EK^2 - EA^2 = AC^2$,

therefore $AK \times KH = AC^2$,

that is, the figure FL = the figure AD

Add to each KC,

then the figure KL = the figure KD,

that is, the square on AK is equal to the rectangle $AB \times BK$

Q E D

EXERCISES

1 Construct a square double of a given square

2 Construct a square equal to two, or three, or any number of given squares

3. Divide a straight line into two parts, such that the square of one of the parts may be half the square on the whole line.

4. Given the base, area, and one of the angles at the base, construct the triangle.

5. Find the locus of a point which moves so that the sum of the squares of its distances from two given points is constant.

We subjoin a few problems and theorems as miscellaneous exercises in the Geometry of angles, lines, triangles, parallelograms, and the equality of areas.

Miscellaneous Theorems and Problems.

1. Prove that the acute angle between the bisectors of the angles at the base of an isosceles triangle is equal to one of the angles at the base of the triangle.

2. Find a point equally distant from three given straight lines

3. If the diagonals of a quadrilateral bisect one another and are equal to one another, the figure will be a rectangle.

4. If the diagonals of a quadrilateral bisect one another at right angles and are also equal, the figure will be a square.

5 If ABC is a triangle, AB being greater than AC, and a point D in AB be taken such that $AD = AC$, prove that the angle BCD is equal to half the difference of the angles ABC, ACB

6 If $ABCD$ is a parallelogram, and $AE = CF$ are cut off from the diagonal AC, then $BEDF$ will be a parallelogram

7 If $AA' = CC'$ be cut off from the diagonal AC, and $BB' = DD'$ from the diagonal BD of a parallelogram, then will $A'B'C'D'$ be also a parallelogram

8 If $AA' = BB' = CC' = DD'$ be cut off from the sides of the parallelogram $ABCD$ taken in order, then will $A'B'C'D'$ be also a parallelogram

9. ABC is a triangle, and through D, the middle point of AB, DE, DF are drawn parallel to the sides BC, AC, to meet them in E, F Shew that EF is parallel to AB

10 Through a given point to draw a line such that the part of it intercepted between two parallel lines shall have a given length

11. To describe a rhombus equal to a given parallelogram, having its side equal to the longer side of the parallelogram

12 Shew that the diagonal of a rectangle is longer than any other line whose extremities are on the sides of the rectangle

13 From the extremities of the base of an isosceles triangle straight lines are drawn perpendicular to the opposite sides, the angles made by them with the base are equal to half the vertical angle.

14. D is the middle point of the side AC of a triangle ACB, and any parallel lines BE, DF are drawn to meet AC, AB (or BC) in E and F, shew that EF divides the triangle into two equal areas.

15. If every pair of alternate sides of a convex figure of five sides be produced to meet, so as to form a five-rayed star, prove that the angles so formed will be together equal to two right angles.

Extend this to the case of a polygon of n sides.

16. Of all triangles having the same base and area, that which is isosceles has the least perimeter.

17. The area of a rhombus is equal to half the rectangle constructed on the two diameters of the rhombus.

18. If two opposite sides of a quadrilateral are parallel, and their points of bisection joined, the quadrilateral will be bisected.

19. If two opposite sides of a parallelogram be bisected, and lines be drawn from these two points of bisection to the opposite angles, these lines will be parallel, two and two, and will trisect both diagonals.

20. The sum of the squares described on the sides of a rhombus is equal to the squares described on its diameters.

21. From the sides of the triangle ABC, AA', BB', CC', are cut off each equal to two-thirds of the side from which it is cut. Shew that the triangle $A'B'C'$ is one-third of the triangle ABC.

22. BCD... are points on the circumference of a circle, A any point not the centre of the circle. Shew that of the lines AB, AC, AD.. not more than two can be equal.

23 Find the locus of a point, such that the sum of the squares on its distances from two given points is equal to the square on the distance between the two points

24. If m and n are any numbers, and lines be taken whose lengths are m^2+n^2, m^2-n^2 and $2mn$ units respectively, shew that these lines will form a right-angled triangle. Give examples of these triangles

25. Through two given points on opposite sides of a straight line draw two straight lines to meet in that line, so that the angle which they form shall be bisected by that line

26 Through a given point draw a line such that the perpendiculars on it from two given points may be equal

27 Find points D, E in the equal sides AB, AC of an isosceles triangle ABC, such that $BD=DE=EC$

28 If one angle of a triangle is equal to the sum of the other two, the greatest side is double of the distance of its middle point from the opposite angle

29 Find the locus of a point, given the sum or difference of its distances from two fixed lines

30 Given two points and a straight line of indefinite length, construct an equilateral triangle so that two of its sides shall pass through the given points, and the third shall be in the given straight line

31. Construct an isosceles triangle having the angle at the vertex double of the angles at the base.

32 ABC is a triangle, AB greater than BC, BD bisects the base AC, and BE the angle ABC Prove (1) that ADB is an obtuse angle, (2) that ABD is less than DBC, and (3) that BE is less than BD

33 Bisect a triangle by a line passing through a point in one of its sides.

34. If two sides of a triangle be given, its area will be greatest when they contain a right angle

35. Construct a triangle equal to a given quadrilateral figure

36 Bisect a given quadrilateral figure by a line drawn from one of its angular points

37. Bisect a given five-sided figure by a line drawn from one of its angular points.

38. If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.

39 Produce a given straight line to such a distance that the square on the produced part may be double of the square on the given line

40 Produce a given straight line to such a distance that the square on the whole line may be double of the square on the given line.

41. Given two sides and a median, construct the triangle.

42. Divide a straight line into two parts such that the square on one part may be four times the square on the other.

43 From B, one of the angles of a triangle ABC, a perpendicular BD is let fall on AC Shew that the difference of the squares on AB, BC is equal to the difference of the squares on AD, DC.

44 AC one of the sides of a triangle ABC is bisected in D: and BD joined Shew that the squares on AB and BC together are equal to twice the square on BD, and twice the square on AD

45 Produce a given line AB to P so that $AP \cdot BP = AB^2$.

46 $ABCD$ is the diameter of two concentric circles, P, Q any points on the outer and inner circles respectively Prove that $BP^2 + CP^2 = AQ^2 + DQ^2$

47. Given a polygon of n sides to construct an equal polygon of $(\overline{n-1})$ sides Hence construct a rectangle equal to any given rectilineal figure.

48 Prove that the squares on the diagonals of any parallelogram are together equal to the squares on its sides

49 O is the point of intersection of the diagonals of a square $ABCD$, and P any other point whatever Prove that $AP^2 + BP^2 + CP^2 + DP^2 = 4OA^2 + 4OP^2$.

50 Given the base, difference of sides, and difference of the angles at the base, construct the triangle

51 If from one of the acute angles of a right-angled triangle a line be drawn to the opposite side, the squares on that side and the line so drawn are together equal to the squares on the segment adjacent to the right angle and on the hypotenuse

52. Find the locus of the middle point of a line drawn from a given point to meet a given line

53 If from the right angle C of a right-angled triangle ABC straight lines be drawn to the opposite angles of the square on AB, the difference of the squares on these two lines will equal the difference of the squares on AC and BC

54. AB is divided into two unequal parts in C and equal parts in D; shew that the squares on AC and BC are greater than twice the rectangle $AC \times CB$ by four times the square on CD.

55. In any right-angled triangle the square on one of the sides containing the right angle is equal to the rectangle contained by the sum and difference of the other two sides

56. In any isosceles triangle ABC, if AD is drawn from A the vertex to any point D in the base, shew that

$$AB^2 = AD^2 + BD \cdot DC.$$

57. Prove that four times the sum of the squares on the medians of a triangle is equal to three times the sum of the squares on the sides of the triangle.

A median of a triangle is the line drawn from an angle to the point of bisection of the opposite side.

58 The square on the base on an isosceles triangle is double the rectangle contained by either side, and the projection on it of the base.

59. The squares on the diagonals of a quadrilateral are double of the squares on the sides of the parallelogram formed by joining the middle points of its sides.

60. Hence shew that they are also double of the squares on the lines which join the points of bisection of the opposite sides of the quadrilateral

61. The squares on the diagonals of a quadrilateral are together less than the squares on the four sides by four times the square on the line joining the points of bisection of the diagonals

62. In any quadrilateral figure the lines which join the middle points of opposite sides intersect in the line which joins the middle point of the diagonals, and bisect one another at that point.

63 The locus of a point which moves so that the sum of the squares of its distances from three given points is constant is a circle.

BOOK III.

THE CIRCLE.

SECTION I.

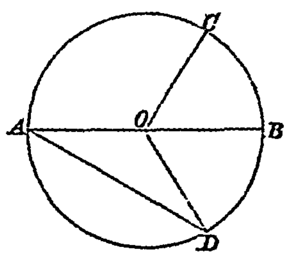
ELEMENTARY PROPERTIES.

A *circle* is a plane figure contained by one line, which is called the circumference, and is such that all the lines drawn from a certain point within the figure to the circumference are equal to one another. This point is called the centre of the circle.

A straight line drawn to the circumference from the centre is called a *radius* of the circle.

A straight line drawn through the centre and terminated both ways by the circumference is called a *diameter* of the circle.

Def. 1. An *arc* is a part of a circumference



Def. 2. A *chord* of a circle is the straight line joining any two points on the circumference. When the arcs into which the chord divides the circumference are unequal, they are called the *major* and *minor* arcs respectively. Such arcs are said to be *conjugate* to one another.

Def 3 A *segment* of a circle is the figure contained by a chord and either of the arcs into which the chord divides the circumference The segments are called *major* and *minor* segments according as the arcs that bound them are major or minor arcs.

Def 4 The *conjugate* angles formed at the centre of a circle by two radii are said to *stand upon* the conjugate arcs opposite them intercepted by the radii, the major angle upon the major arc, and the minor angle upon the minor arc

Def 5 A *sector* is the figure contained by an arc and the radii drawn to its extremities The *angle of the sector* is the angle at the centre which stands upon the arc of the sector

Def 6 Circles that have a common centre are said to be *concentric*.

The following properties of the circle are immediate consequences of Book I Def 8

(*a*) A circle has only one centre

(*b*) A point is within, on, or without the circumference of a circle, according as its distance from the centre is less than, equal to, or greater than the radius

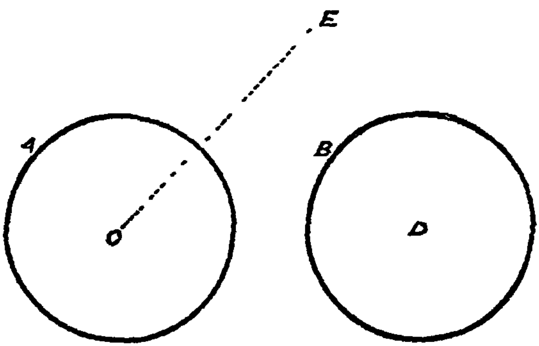
(*c*) The distance of a point from the centre of a circle is less than, equal to, or greater than the radius, according as the point is within, on, or without the circumference

THEOREM I

Circles of equal radii are identically equal.

Part. En. Let A and B be circles of equal radii; it is required to prove that they are identically equal.

Proof. Let their centres be C and D. Place the circle B upon the circle A so that the point D falls upon the point C, and take any point E outside both circles and



join CE. Then since all radii of the same circle are equal, and the circles are of equal radii;

therefore the distances from C along CE to the circumferences of the two circles are the same,

therefore the circumferences cut the line CE in the same point

Similarly they cut every line through C in the same point, and therefore coincide altogether, and the two circles are identically equal.

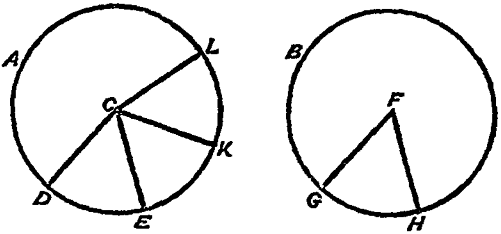
COR. *Two (different) circles whose circumferences meet one another cannot be concentric*.*

* Euclid, III 5.

THEOREM 2

*In the same circle, or in equal circles, equal angles at the centre stand on equal arcs, and of two unequal angles at the centre the greater angle stands on the greater arc**

Part En Let A and B be two equal circles, and DCE, GFH two angles at their centres C and F, standing upon the arcs DE, GH respectively, it is required to



prove that if the angle DCE be equal to the angle GFH, the arc DE will also be equal to the arc GH, and if the angles be not equal then the greater angle will stand upon the greater arc

Proof Place the circle B upon the circle A so that the point F falls upon the point C, and the bounding lines of the angle GFH upon those of the equal angle DCE, then will the two circles coincide, and the points G and H will fall on the points D and E because the circles are of equal radii, (III 1)

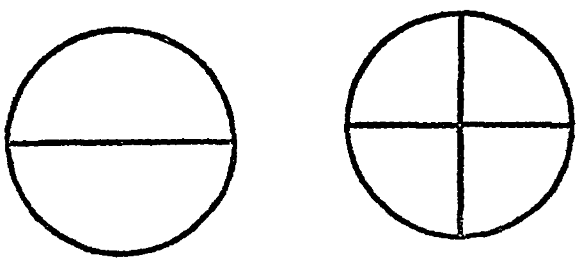
and the circumferences coinciding, and G and H falling on D and E, then will the arc GH fall on and coincide with the arc DE, and be therefore equal to it

* Euclid, III 26.

Again, if the angles DCE, GFH be not equal, let DCE be the greater. Then it is possible to place the circle B upon the circle A so that the point F falls on the point C, and the bounding lines of the angle GFH fall within the bounding lines of the greater angle DCE; and therefore the minor arc GH forms a part of, and therefore is less than, the minor arc DE.

And further, since any angle KCL at the centre of the circle A, equal to GFH, stands upon an arc equal to GH, therefore the arc DE is greater than, equal to, or less than the arc KL in the same circle, according as the angle DCE is greater than, equal to, or less than the angle KCL.

COR. 1. *Sectors of the same or of equal circles which have equal angles are equal; and of two such sectors which have unequal angles that which has the greater angle is the greater.*



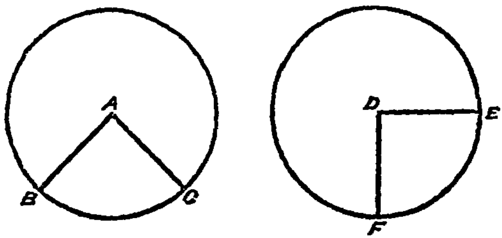
COR. 2. *A diameter of a circle divides it into two equal parts which are called semicircles, and two diameters at right angles to one another divide the circle into four equal parts which are called quadrants.*

Proof. The quadrants are sectors whose angles are right angles and are therefore equal to one another, by Cor 1; and the semicircles are sectors whose angles are angles of two right angles, and are therefore equal to one another

Def. 7. The former are called *semicircles*, and the latter are called *quadrants*

THEOREM 3.

In the same circle, or in equal circles, equal arcs subtend equal angles at the centre; and of two unequal arcs the greater subtends the greater angle at the centre.*



Part En. Let the circles whose centres are A and D be equal, and let BC, FE be arcs;

then it is required to prove that if the arc BC be equal to the arc FE, the angle BAC will be equal to the angle FDE, and if the arc BC be greater than the arc FE, the angle BAC will be greater than the angle FDE

Proof. First, let the arc BC be equal to the arc FE, then the angle BAC will be equal to the angle FDE

For the angle BAC is either equal to the angle FDE or unequal to it;

* Euclid, III 27.

but if the angle BAC were unequal to the angle FDE, then the arc BC would be unequal to the arc FE; (III 2)

but it is not,

and therefore the angle BAC is equal to the angle FDE.

Secondly, let the arc BC be greater than the arc FE, then the angle BAC will be greater than the angle FDE.

For the angle BAC is either greater than, equal to, or less than the angle FDE;

But the angle BAC is not equal to the angle FDE, for then the arc BC would be equal to the arc FE, (III. 2) but it is not:

nor is the angle BAG less than the angle FDE;

for then the arc BC would be less than the arc FE, (III. 2)

but it is not;

therefore the angle BAC is greater than the angle FDE.

Obs (1) This theorem affords an excellent example of an application of the *rule of conversion* (p 3) It must be observed that

Theorem 2 forms in fact a group of theorems in which it is demonstrated that (see figure to Theorem 3),

(1) if the angle A is greater than the angle D, the arc BC is greater than the arc FE;

(2) if the angle A is equal to the angle D, the arc BC is equal to the arc FE;

(3) if the angle A is less than the angle D, the arc BC is less than the arc FE.

Now of the Hypotheses of these theorems one must be true;

for the angle A must be greater than, equal to, or less than the angle D;

and of the Conclusions no two can be true at the same time, for the arc BC cannot be both greater than, and equal to, the arc FE, therefore the *rule of conversion* applies to this group of Theorems, that is, their *converses,* which form Theorem 3, are true In the text this proof is given in detail But when these conditions are true of a group of theorems their converses are always necessarily true, and no special proof is necessary

Obs (2) This theorem might also be proved by direct superposition

COR. *Equal sectors of the same or of equal circles have equal angles, and of two unequal sectors the greater has the greater angle.*

This may be proved as in the theorem, or by the rule of conversion, (which applies to the group of Theorems contained in Theorem 2, Cor 1), or by direct superposition

EXERCISES ON SECTION I

1 What arc of a circle is equal to its conjugate arc? What arc is half its conjugate arc?

2 What segment of a circle is also a sector of a circle?

3 Prove that if two circles cut one another they cannot have the same centre.

4 Divide a circle into eight equal parts by radii

5 If there be two sectors of equal circles, and the angle of the first is any multiple of the angle of the second, prove that the area of the first is the same multiple of the area of the second.

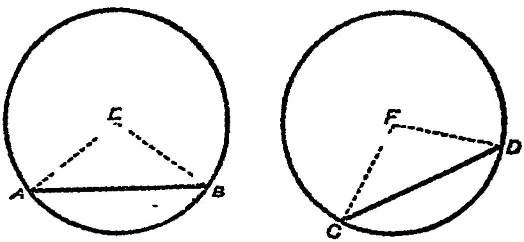
SECTION II.

Chords

Theorem 4.

*In the same circle, or in equal circles, equal arcs are subtended by equal chords; and of two unequal minor arcs, the greater is subtended by the greater chord**

Part. *En* Let AB, CD be equal arcs of the same or of equal circles;



it is required to prove that the chord AB is equal to the chord CD.

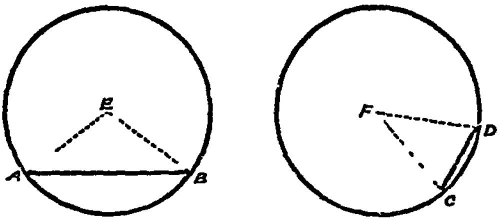
Proof. Let E, F be the centres of the circles: join AE, BE, CF, DF.

Then because the arc AB is equal to the arc CD; (Hyp.)
therefore the angle AEB is equal to the angle CFD; (III 3.)
and because in the two triangles AEB, CFD, the two sides

* Euclid, III. 29.

AE, EB are equal to the two CF, FD, and the contained angle AEB is equal to the contained angle CFD,
therefore the base AB is equal to the base CD (I 5)

Part En Again, let the minor arc AB be greater than the minor arc CD, it is required to prove that the chord AB is greater than the chord CD.



Proof Because the arc AB is greater than the arc CD, therefore the angle AEB is greater than the angle CFD, (III 3)

and because in the triangles AEB, CFD, the two sides AE, EB are equal to the two CF, FD, but the contained angle AEB is greater than the contained angle CFD,

therefore the base AB is greater than the base CD (I 14.)

COR *In the same circle, or in equal circles, of two unequal major arcs the greater is subtended by the less chord*

Obs Since AB, CD are minor arcs, the angles AEB, CFD are less than two right angles This has been assumed in the proof in treating AEB, CFD as triangles

THEOREM 5

*In the same circle, or in equal circles, equal chords subtend equal major and equal minor arcs, and of two unequal chords the greater subtends the greater minor arc and the less major arc**

* Euclid, III 28

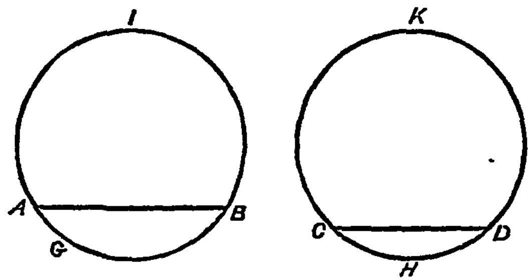
Part En Let *AB*, *CD* be equal chords in the same or equal circles, it is required to prove that the arcs *AB*, *CD* are equal.

Proof. For if the minor arcs *AB*, *CD* were unequal, one of them would be the greater,
and therefore the chord *AB* would be unequal to the chord *CD*. (III 4)

But it is not, for the chords are equal (Hyp)

Therefore the minor arcs *AB*, *CD* are also equal.

Again, let the chord *AB* be greater than the chord *CD*,



it is required to prove that the minor arc *AGB* is greater than the minor arc *CHD*.

Proof. For the minor arc *AGB* is either greater than, equal to, or less than the minor arc *CHD*;

But the minor arc *AGB* is not equal to the minor arc *CHD*,
for then the chord *AB* would be equal to the chord *CD*. (III 4)

But it is not,

Nor is the minor arc *AGB* less than the minor arc *CHD*, for then the chord *AB* would be less than the chord *CD*. (III 4.)

But it is not,

therefore the minor arc AGB is greater than the minor arc CHD,

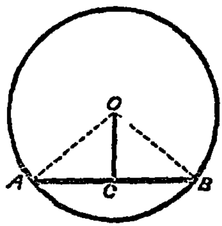
and therefore also the major arc AIB is less than the major arc CKD

Obs As before, the rule of conversion applies to the groups of theorems enunciated in Theorem 4 and Cor, and their converses form Theorem 5

THEOREM 6

The straight line drawn from the centre of a circle to the middle point of a chord is perpendicular to the chord

Part En Let the straight line OC be drawn from the centre O of a circle to the middle point C of a chord AB, it is required to prove that OC is perpendicular to AB



Proof. Join OA, OB

Then because in the triangles OAC, OBC, the three sides of the one are respectively equal to the three sides of the other,

therefore the angles OCA, OCB opposite to the equal sides OA, OB are equal (I 15), and are therefore right angles,

that is, OC is perpendicular to AB*

THEOREM 7.

The straight line drawn from the centre of a circle perpendicular to a chord bisects the chord†.

* Euclid, III 3, Part 1 † Euclid, III 3, Part 2

Part. En. Let the straight line OC drawn from the centre O of a circle be perpendicular to the chord AB, it is required to prove that OC bisects AB

Proof. Because in the right-angled triangles ACO, BCO,

the hypotenuse AO is equal to the hypotenuse BO,

and the side OC is common,

therefore the triangles are equal in all respects, (I 20, Cor)

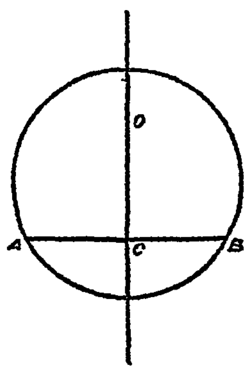
and therefore AC is equal to CB, and OC bisects AB.

THEOREM 8

The straight line drawn perpendicular to a chord of a circle through its middle point passes through the centre of the circle.*

Part En. Let AB be a chord of a circle, bisected in C, and let CO be drawn at right angles to AB, it is required to prove that CO passes through the centre.

Proof. Because CO bisects AB at right angles, (Hyp) therefore CO is the locus of points equidistant from A and B (p 72) But the centre of the circle is equidistant from A and B



therefore CO passes through the centre.

COR. *The locus of the centres of all circles that pass through two given points is the straight line that bisects at right angles the line joining those points.*

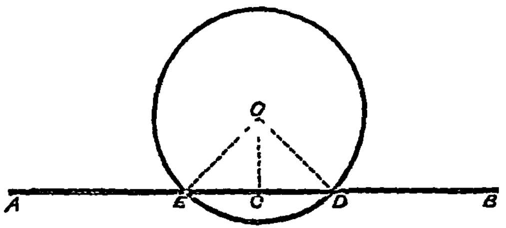
* Euclid, III 1. Cor.

Remark On Theorems 6, 7, 8 it may be remarked that if any one of them be proved directly, the other two follow from applications of the Rule of Identity For example, if Theorem 7 be proved directly and it be required to demonstrate Theorem 6, we may proceed as follows —Of straight lines through the centre there can be *but one* to the middle point of the chord, and *but one* perpendicular to it; and inasmuch as by Theorem 7, the one that is perpendicular to the chord is also the one that bisects it, it follows by the Rule of Identity that the one that bisects it is also the one that is perpendicular to it.

THEOREM 9

A straight line cannot meet the circumference of a circle in more than two points.

Part En Let AB be a straight line, O the centre of a circle,



It is required to prove that the straight line AB does not meet the circumference of the circle in more than two points

Proof Draw OC perpendicular to AB

Let D be one of the points of intersection of the straight line, and circle, join OD,

and at the point O in the line OC on the side of OC remote from D, make the angle COE equal to COD, and let OE meet the straight line AB in E.

Then since OE, OD are obliques making equal angles with the perpendicular OC,

therefore OD is equal to OE, (I 19)

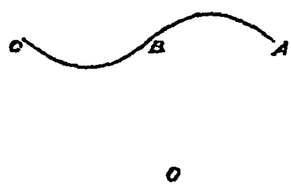
and therefore E is on the circumference of the circle,

and all lines drawn from O to AB other than OE are greater or less than OD, that is no point on AB other than D and E is on the circle.

Cor. *A chord of a circle lies wholly within the circle.*

Obs. Hence a circle is everywhere concave to its centre.

For the test of the concavity of an arc of a curve to a given point is that, if any two points in the arc be taken, the chord joining those points shall be cut by every line drawn from the given point to a point on that arc.



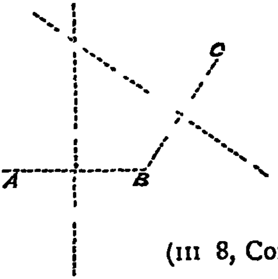
Thus in the figure the arc AB is concave to O, and the arc BC is not concave to O, and is said to be *convex* to it

THEOREM 10

One circle, and only one, can be drawn through any three points not in the same straight line.

Part. En. Let A, B, C be three points, not in the same straight line;

it is required to prove that one circle, and only one, can be drawn to pass through *A*, *B*, *C*



Proof Because the locus of the centres of all circles that pass through *A* and *B* is a straight line that bisects *AB* at right angles, (III 8, Cor)

and the locus of the centres of all circles that pass through *B* and *C* is the straight line that bisects *BC* at right angles, therefore the centre of a circle that passes through *A*, *B* and *C* is a point common to these two straight lines.

Now these straight lines will intersect, for if they did not they would be parallel, and therefore *A*, *B*, and *C* would be in one straight line,

and they can intersect in only one point. (Ax 2)

therefore one circle, and only one circle, can be drawn to pass through *A*, *B* and *C*

COR 1. *Two circles that have three points in common coincide wholly*

Hence a circle is named by three letters at points on its circumference.

COR 2 *Two circles cannot meet one another in more than two points**.

For if they had three points in common they would coincide wholly

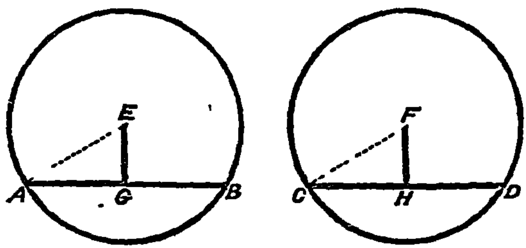
COR. 3 *If from any point within a circle more than two equal straight lines are drawn to the circumference, that point is the centre of the circle†.*

* Euclid, III 10 † Euclid, III 9

THEOREM 11.

*In the same circle, or in equal circles, equal chords are equally distant from the centre; and of two unequal chords the greater is nearer to the centre than the less**.

First. Part. En. Let AB, CD be equal chords in the same, or in equal circles, whose centres are E, F: and let



EG, FH be perpendiculars from E, F on AB, CD respectively; it is required to prove that EG is equal to FH.

Proof. Join EA, FC.

Then because EG, FH are perpendiculars on the chords from the centres,

therefore they bisect the chords: (III. 7)

and, because the chords are equal, (Hyp)

therefore AG is equal to CH

and therefore in the right-angled triangles AGE, CHF,

the hypotenuse AE is equal to the hypotenuse CF, (Hyp.)

* Euclid, III. 14, 15 Part I.

and one side AG is equal to one side CH

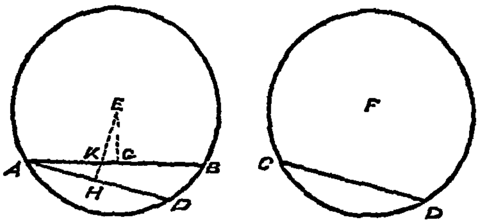
therefore the side EG is equal to FH. (I. 20, Cor)

that is, the chords AB, CD are equally distant from the centre.

Again,

Part En. Let AB, CD be unequal chords of the same, or of equal circles, of which AB is the greater,

it is required to prove that AB is nearer to the centre than CD.



Proof. Because the chord AB is greater than the chord CD,

therefore the minor arc AB is greater than the minor arc CD. (III. 5)

Place the circles on one another, with E as their common centre, (III 1)

so that the point C falls on A, and the point D on the minor arc AB; and let fall the perpendiculars EG, EH on the chords AB, AD

Let EH cut AB in K

then because EG is the perpendicular on AB from E,

therefore EG is less than EK. (I. 19)

and EK is less than EH:

therefore EG is less than EH.

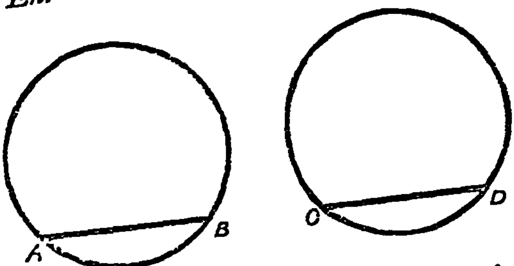
that is, AB is nearer to the centre than AD or CD

Obs The first part of this theorem may be proved by superposition.

THEOREM 12.

In the same circle, or in equal circles, chords that are equally distant from the centre are equal, and of two chords unequally distant, the one nearer to the centre is the greater.*

Part En. Let AB, CD be chords of the same or of



equal circles, equally distant from the centre · it is required to prove that AB is equal to CD.

Proof. For if AB were unequal to CD, one of them would be the greater, and would therefore be nearer to the centre than the less (III 11

But it is not, for they are equally distant from the centre, (Hyp

therefore AB is equal to CD.

Again,

Part. En Let AB be nearer to the centre than CD it is required to prove that AB is greater than CD.

* Euclid, III 14, 15 Part 2

Proof. For AB must be either greater than CD, or equal to CD, or less than CD

But AB is not equal to CD,
for then AB and CD would be equally distant from the centre. (III 11)

But they are not·

Nor is AB less than CD,
for then AB would be further from the centre than CD, (III 11)
but it is not,
therefore AB is greater than CD

COR *The diameter is the greatest chord in a circle*

Obs This theorem follows logically from Theorem 11 For the group of theorems in Theorem 11 is such that of their hypotheses *one must be true*, that is, the chord AB must be greater than, equal to, or less than the chord CD, *and of the conclusions*, AB is at a less, equal, or greater distance from the centre than CD, *no two can be true at the same time*, therefore the rule of conversion is applicable; and Theorem 11 contains the converse theorems thus established

It may also be remarked that this Theorem may be proved by superposition

EXERCISES ON SECTION II

1 Given a triangle ABC to find the centre of the circle that passes through A, B, and C

2 If two equal chords intersect one another, the segments of the one are equal to the segments of the other respectively.

3 Two chords cannot bisect one another unless both pass through the centre

4 Given a curve, to ascertain whether it is an arc of a circle or not.

5. If a straight line cut two concentric circles, the parts of it intercepted between the two circumferences will be equal.

6 Perpendiculars are let fall from the extremities of a diameter on any chord, or any chord produced; shew that the feet of the perpendiculars are equally distant from the centre.

7. The locus of the points of bisection of parallel chords of a circle is the diameter at right angles to those chords.

8. If a diameter of a circle bisects a chord which does not pass through the centre, it will bisect all chords which are parallel to it.

9 AB and CD are unequal parallel chords in a circle; prove that AC and BD, and likewise AD and BC, intersect on the diameter perpendicular to AB and CD, or that diameter produced, and are equally inclined to that diameter.

What will be the case if AB and CD are equal?

SECTION III.

Angles in Segments

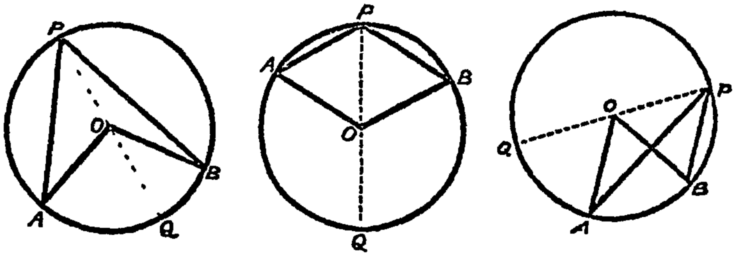
Def 8. The angle formed by any two chords drawn from a point on the circumference of a circle is called an angle *at the circumference*, and is said to *stand upon the arc* between its arms

Def 9 An angle contained by two straight lines drawn from a point in the arc of a segment to the extremities of the chord is called an *angle in the segment*.

Theorem 13

An angle at the circumference of a circle is half the angle at the centre standing on the same arc.

Part En. Let AB be an arc, O the centre, P any



point on the circumference of a circle; it is required to prove that the angle APB is half of the angle AOB standing on the same arc.

Proof. Join PO, and produce it to Q.

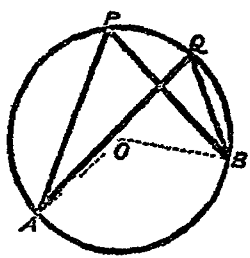
Then because OA is equal to OP;
therefore the angle OAP is equal to the angle OPA: (I. 6)
but the exterior angle AOQ is equal to the two interior and opposite angles OAP and OPA, (I. 25)
therefore the angle AOQ is double of the angle OPA.
Similarly the angle QOB is double of the angle OPB

Hence in figs (1 and 2) the sum, or (in fig. 3) the difference of the angles AOQ, QOB is double of the sum or difference of OPA and OPB,
that is, the angle AOB is double of the angle APB;
and therefore the angle APB is half of the angle AOB on the same arc*.

THEOREM 14.

Angles in the same segment of a circle are equal to one another.

Part. En. Let APB, AQB be angles in the same segment $APQB$,



it is required to prove that the angle APB is equal to the angle AQB.

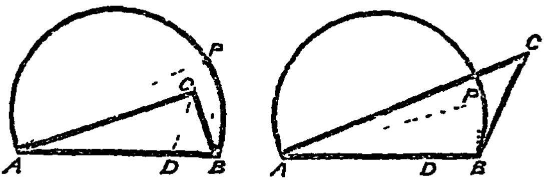
Proof. Take O the centre, and join AO, BO.

Then because the angles APB, AQB at the circumference are each of them half the angle AOB at the centre on the same arc, (III. 13)
therefore the angle APB is equal to the angle AQB†.

* Euclid, III 20 † Euclid, III. 21.

COR. *The angle subtended by the chord of a segment at a point within it is greater than, and at a point outside its segment on the same side of the chord as the segment, is less than, the angle in the segment*

Part Ex Let APB be a segment of a circle, and C a point on the same side of AB as the segment, it is required to prove that the angle



ACB is greater or less than the angle in the segment APB according as C is within or without the segment

Proof Take any point D in AB and join CD, and let CD (produced if necessary) meet the curved boundary of the segment in P Join PA, PB

Then if C is within the segment APB it is evidently within the triangle APB, and therefore the angle ACB is greater than the angle APB (I 13)

Again if C is without the segment APB, P is evidently within the triangle ACB, and therefore the angle ACB is less than the angle APB (I. 13)

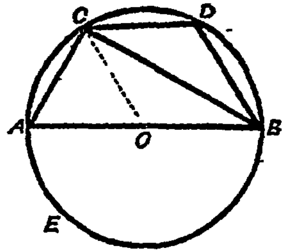
Remark From this theorem and its corollary we learn that *the locus of a point on one side of a given straight line at which that straight line subtends a constant angle is an arc of a circle of which that line is the chord*

THEOREM 15

*The angle in a segment is greater than, equal to, or less than a right angle, according as the segment is less than, equal to, or greater than a semicircle**

* Euclid, III 31.

Part. En. Let AB be a diameter of a circle, cutting off the semicircle ADB; and let any other chord BC divide the circle into the segment BDC less than a semicircle, and BEC greater than a semicircle;



it is required to prove that the angle in the segment BDC less than a semicircle is greater than a right angle;

and that the angle in the semicircle BDA is equal to a right angle; and that the angle in the segment BEC greater than a semicircle is less than a right angle.

Proof. Let O be the centre; join CO.

Then the angle in the segment CDB is half the angle COB subtended at the centre by the same arc BEC. (III 13)

But this is a reflex angle, and is greater than two right angles;

therefore the angle in the segment CDB is greater than one right angle.

Again, the angle in the semicircle ADB is half the angle AOB upon the same arc AEB. (III. 13)

But the angle AOB is equal to two right angles;

therefore the angle in the semicircle is equal to a right angle.

Lastly, the angle in the segment CEB is half the angle COB. (III. 13)

But the angle COB is less than two right angles;

therefore the angle in the segment CEB is less than a right angle.

THEOREM 16

A segment is less than, equal to, or greater than a semicircle according as the angle in it is greater than, equal to, or less than a right angle

Proof According as the angle in the segment, that is at the circumference, is greater than, equal to, or less than a right angle, the angle at the centre will be greater than, equal to, or less than two right angles;

that is, the segment is less than, equal to, or greater than a semicircle.

Alternative Proof For *of the hypotheses* that a segment is either greater than, equal to, or less than a semicircle, *one must be true*, and *of the conclusions* proved in Th 15 that the angle in that segment is either less than, equal to, or greater than a right angle, *no two can be true at the same time*,

therefore the converses of the theorems in Th 15 are true,

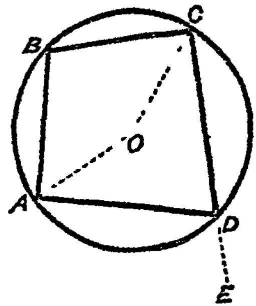
that is, according as the angle in a segment is less than, equal to, or greater than a right angle, that segment is greater than, equal to, or less than a semicircle

THEOREM 17

*The opposite angles of a quadrilateral inscribed in a circle are supplementary**.

* Euclid, III 22.

Part 2n. Let $ABCD$ be a quadrilateral inscribed in a circle; it is required to prove that its opposite angles ABC, CDA are supplementary.



Proof. Take O the centre, and join AO, CO.

Then the angles ABC, CDA are respectively the halves of the angles made by AO, OC at the centre O. (III. 13)

But the sum of the angles at the centre O is four right angles: (I 4, COR)

therefore the sum of the angles ABC, CDA is two right angles;

that is, the angles ABC, CDA are supplementary.

COR. 1. *The exterior angle of a quadrilateral inscribed in a circle is equal to the interior opposite angle*

For if CD is produced to E, the exterior angle ADE is supplementary to ADC, and is therefore equal to ABC.

COR 2. *If the opposite angles of a quadrilateral are supplementary, the quadrilateral can be inscribed in a circle.*

EXERCISES ON SECTION III.

1. Prove that the lines which join the extremities of equal arcs in a circle are either equal or parallel.

2. If two opposite sides of a quadrilateral inscribed in a circle are equal, prove that the other two are parallel.

3 AB, CD are chords of a circle which cut at a constant angle Prove that the sum of the arcs AC, BD remains constant, whatever may be the position of the chords

4 If the diameter of a circle be one of the equal sides of an isosceles triangle, prove that its circumference will bisect the base of the triangle

5 Circles are described on two sides of a triangle as diameters Prove that they will intersect on the third side or third side produced

6 Any number of chords of a circle are drawn through a point on its circumference · find the locus of their middle points

7 If through any point, within or without a circle, lines are drawn to cut the circle, prove that the locus of the middle points of the chords so formed is a circle

8 In any inscribed hexagon the sum of any three alternate angles is equal to four right angles

SECTION IV. A.

TANGENTS (*treated directly*).

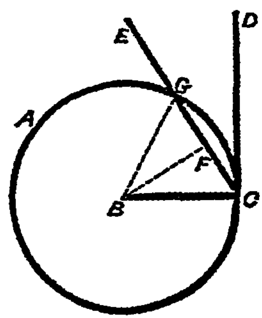
Def. 10. A secant is a straight line of unlimited length which meets the circumference of a circle in two points.

THEOREM 18.

*Every straight line through a point on the circumference of a circle meets it in one other point, except the straight line perpendicular to the radius at the point**.

Part. En. Let A be a circle, B its centre, and BC a radius; and let CD be a line through C perpendicular to the radius BC, and CE any other line;

it is required to prove that CE meets the circle in one point other than C, and that CD does not.



Proof. Because BC is perpendicular to CD,

therefore BC is the shortest line from B to the line CD: (I. 19)

therefore every point in CD other than C is at a distance from B greater than BC, that is than the radius of the circle.

Therefore no point in CD other than C is on the circumference.

* Euclid, III 16.

Again, from B draw BF perpendicular to CE, and BG making an angle with BF, on the side remote from C, equal to CBF, and meeting CE in G.

Then because BC and BG are straight lines from B to the straight line CE making equal angles with the perpendicular BF upon it, they are equal; (I 19)

that is, BG is equal to the radius of the circle,

and therefore G lies upon the circumference; that is, the line CE meets the circle again in G.

Def 11. A straight line which, though produced indefinitely, meets the circumference of a circle in one point only is said to *touch*, or to be a *tangent* to, the circle

Def 12 The point at which a tangent meets the circumference is called the *point of contact*.

The following are immediate consequences of Theorem 18

(*a*) One and only one tangent can be drawn to a circle at a given point on the circumference.

(*b*) The tangent to a circle is perpendicular to the radius drawn to the point of contact.

(*c*) The centre of a circle lies in the perpendicular to the tangent at the point of contact.

(*d*) The straight line drawn from the centre perpendicular to the tangent passes through the point of contact.

Obs On the relative position of a straight line and a circle.

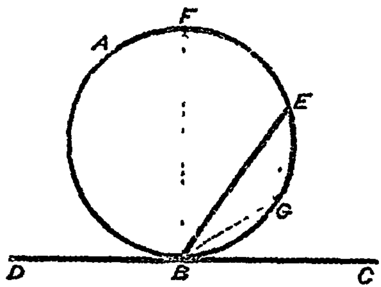
A straight line will cut a circle, touch it, or not meet it at all, according as its distance from the centre is less than, equal to, or greater than the radius

The several converses of these statements follow by the Rule of Conversion

THEOREM 19.

*Each angle contained by a tangent and a chord drawn from the point of contact is equal to the angle in the alternate segment of the circle**.

Part. En Let DBC be a tangent to the circle A at the point B, and let BE be a line through B meeting the circle again in E,



it is required to prove that the angles contained by DBC and BE are equal to the angles in the alternate segments upon BE.

Proof. Draw BF the diameter through B;
then BF will be at right angles to DC; (Th. 18.)
and join F, E and B to any point G in the minor arc BE.

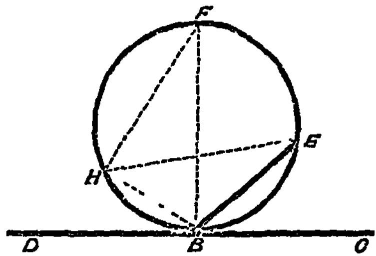
Then because FGB is an angle in a semicircle it is a right angle;

and therefore the angle FGB is equal to the angle FBD, also the angle EGF is equal to the angle EBF in the same segment;

* Eucl III 32.

therefore the whole angle EGB is equal to the whole angle EBD

Again, join F and E and B to any point H in the circumference on the side of BF remote from E Then because FHB is an angle in a semicircle it is a right angle,



therefore the angle FHB is equal to the angle FBC,

also the angle FHE is equal to the angle FBE in the same segment,

therefore the remaining angle EHB is equal to the remaining angle EBC.

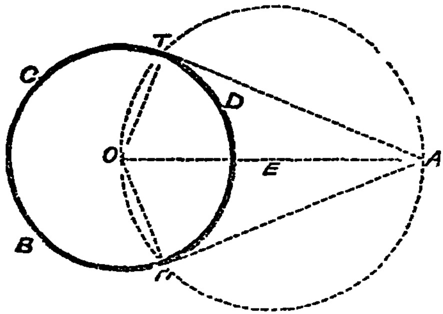
Obs Having proved Th 19 so far as it relates to *either* of the two angles EBC, EBD, its truth as it relates to the *other* follows at once from Th 17, since the angle in the conjugate segment and the remaining angle at B are respectively supplementary to the two equal angles.

THEOREM 20.

Two tangents, and only two, can be drawn to a circle from an external point

Part En Let A be a point external to the given circle BCD, it is required to prove that two, and only two, straight lines can be drawn from A to touch the circle BCD.

Proof. Take O the centre, join OA, bisect it in E, and with centre E and radius EO or EA describe a circle



Then OA will be a diameter of that circle, and each of the portions into which it divides the circumference will cut the circle BCD, because each is a continuous line with one extremity within and one extremity without the circle.

Let them meet it in T and T' respectively Join OT and AT Then because OTA is an angle in a semicircle it is a right angle, (III 15)
therefore TA touches the circle BCD at the point T (III 18)
Similarly AT' touches the same circle at T'

Therefore two straight lines can be drawn from A to touch the circle

Again, there cannot be more than two straight lines drawn from A to touch the circle

For because the angle between the radius and the tangent is a right angle, (III 18)
therefore the point of contact lies on the circle described on AO as diameter (III 16 and III 13 Obs)

But this circle cannot intersect the given circle in more than two points (III 10. Cor 2)

Therefore there cannot be drawn more than two tangents from A to the circle.

COR. *The two tangents drawn to a circle from an external point are equal and make equal angles with the straight line joining that point with the centre*

For let AT, AT' be the two lines touching the circle in T and T' Then because OT is equal to OT',

and OA is common to the two triangles OAT and OAT', and the angles at T and T' are right angles,

therefore the triangles are equal and the angle OAT is equal to the angle OAT', (I 20)

and therefore the tangents from A are equal and make equal angles with OA

SECTION IV B

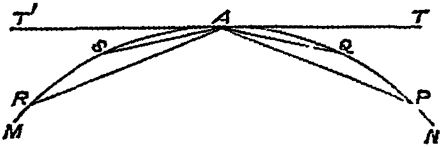
TANGENTS (*treated by the method of limits*)

This may be omitted the first time of reading

There is another light in which we may regard the lines of which we have been speaking in Section IV (A), which is extremely valuable when we come to consider curves other than circles. We shall proceed to give an account of it

Let MAN be a curve, not necessarily a circle, but one which curves in the same direction throughout as you proceed from M towards N Take a line through A meeting the curve at some point P between A and N Then the

nearer P is taken to A the nearer does the line AP approach to a position represented in the figure by the



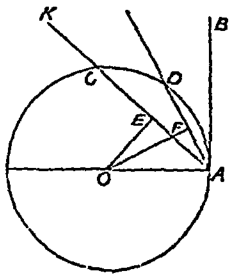
line $T'AT$. So long as P is between A and N it can never quite coincide with the said line TAT', but it can be made to approach as near to it as we please by taking P close enough to A.

Similarly if we take a line through A meeting the curve in some point R between A and M, then the nearer that point lies to A the nearer will the line AR approach the position TAT'. It can never quite coincide with the said line, as long as R is between A and M, but it may be made to approach as near to it as we please by taking R near enough to A.

It may not be easy to see how the line $T'AT$ is to be accurately obtained, but it will easily be seen that there is in general such a line at each point of a curve, and it will be distinguished from other lines drawn through the point by the peculiarity that *it does not cross the curve at that point.* Such a line is said to *touch* the curve at that point, or, more formally,—if a secant of a curve alters its position in such a manner that the two points of intersection continually approach, and ultimately coincide with one another, the secant in its limiting position is said to *touch*, or to be a *tangent* to, the curve, and the point at which the tangent meets the curve is called the *point of contact.*

We shall now investigate the position of the *tangent* to a circle at any specified point, using our newly obtained definition of a *tangent*

Let A be the point and O the centre of the circle, and let AK be a straight line through A, not at right angles to OA, and AB a line through A perpendicular to OA Draw OE perpendicular to AC, then $OE < OA$ (I 19)



∴ E is within the circle,

and AK must meet the circle in some point other than A Let it be C

Now let C move up towards A, then the chord AC will become shorter, and the perpendicular OE, which bisects the chord AC, will approach nearer to coincidence with OA. Hence the line AC will approach nearer to the position of being perpendicular to OA And inasmuch as the chord AC can be made as short as we please, and thus the line OE can be made to approach as near as we please to OA, the line AK can be made to approach as near as we please to the position of AB. *Hence AB is the tangent at A*

And inasmuch as no straight line can meet the circle in more than two points, and the line AB is the limiting position of a secant through A when the other point of intersection has moved up to coincidence with A, it follows that the line AB cannot meet the circle again Hence every straight line through a point on the circumference meets it in one other point, except the straight line perpendicular to the

radius at the point, *and this is the tangent at the point*, which is Theorem 18.

It is evident that we shall arrive at exactly the same result by supposing that the point C is on the other side of OA, and moves up to coincidence with A in the other direction

Def 11 If a secant of a circle alters its position in such a manner that the two points of intersection continually approach, and ultimately coincide with one another, the secant in its limiting position is said to *touch* or to be a *tangent* to, the circle

Def 12. The point in which two points of intersection ultimately coincide is called the *point of contact* and the tangent is said to touch the circle at that point.

Taking this definition of a tangent, Theorem 6 gives us

The straight line drawn from the centre to the point of contact of a tangent is perpendicular to the tangent This is (b) in the last section.

Theorem 7 gives us

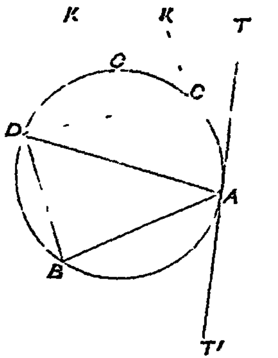
The straight line drawn from the centre perpendicular to a tangent passes through the point of contact This is (d) in the last section

Theorem 8 gives us

The straight line drawn perpendicular to a tangent through its point of contact passes through the centre This is (c) in the last section

Theorem 17, Cor 1, gives us Theorem 19

For if $ABDC$ be a quadrilateral inscribed in a circle and AC be produced to K, then the angle KCD is equal



to the angle ABD, that is, to the angle in the segment ABD Now let C move up to coincidence with A. The angle DCK will remain the same in magnitude, and it finally coincides with DAT, for AK will move up to coincidence with AT the tangent at A, and DC will move up to coincidence with DA Hence the angle DAT is equal to the angle DBA in the alternate segment DBA Similarly we can shew that DAT' is equal to the angle DCA in the alternate segment DCA.

EXERCISES ON SECTION IV

1 Prove that the two tangents drawn to a circle from any external point are equal.

2 If from a point without a circle two tangents AB, AC are drawn, the chord of contact BC will be bisected at right angles by the line from A to the centre.

3 If a circle is inscribed in a right-angled triangle, the excess of the two sides over the hypothenuse is equal to the diameter of the circle

4. If a quadrilateral figure be described about a circle, the sums of the opposite sides will be equal to one another

5. If a six-sided figure be circumscribed about a circle, the sums of the alternate sides will be equal

6 If a quadrilateral figure be described about a circle, the angles subtended at the centre by any two opposite sides are together equal to two right angles.

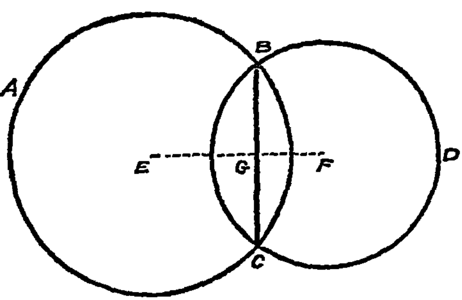
SECTION V.

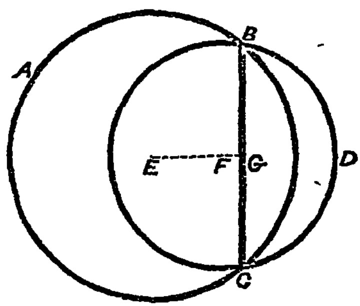
The Relations of Two Circles

Theorem 21.

The straight line which passes through the centres of two circles whose circumferences meet in two points bisects the straight line joining those points, and is at right angles to it.

Part En Let ABC, DBC be two circles intersecting in the points B and C, and let E and F be their centres;



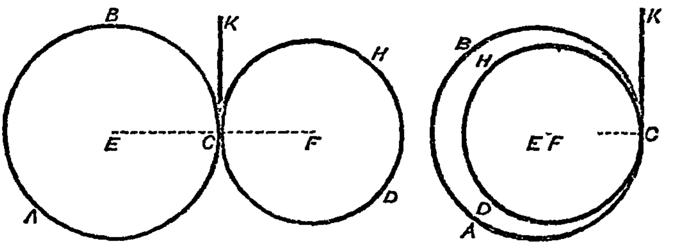


it is required to prove that the straight line EF bisects the common chord BC and is at right angles to it

Proof Bisect BC in G, and join GE, GF Then because BC is a chord of the circle BCD, and FG is a line drawn through the centre F bisecting it,
therefore the angle BGF is a right angle, (III 6)
and because BC is a chord of the circle BAC, and EG is drawn from the centre E bisecting it,
therefore the angle EGB is a right angle;
therefore EG, GF are in one straight line, (I 3)
that is, are in the straight line joining E and F
Therefore the straight line EF bisects the common chord BC and is at right angles to it

THEOREM 22

If the circumferences of two circles meet at a point on the straight line passing through their centres, these circumferences cannot have a second point in common



Let the two circles whose centres are E, F have one point C in common on the straight line passing through their centres, then these circumferences shall not have any other point in common

Proof. For if another point as B were common, then by Th 21, the straight line EF would not pass through E, but bisect CB at right angles

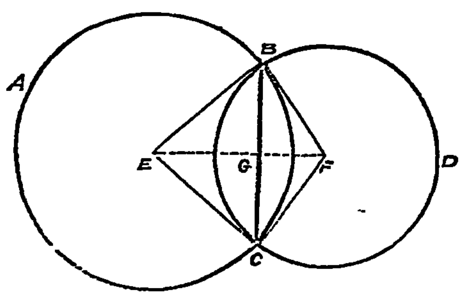
But EF does pass through C;

therefore the circumferences have no other point in common except C.

Def. 13 Two circles whose circumferences meet in one point only are said to *touch* each other, and the point at which they meet is called their *point of contact*

THEOREM 23.

If the circumferences of two circles have one common point not on the line through their centres, they have also another common point



Let the two circles whose centres are E, F have one common point B, not on EF. They shall have also another common point.

Proof From B let fall BG perpendicular to EF, and produce BG to C, making $GC = GB$. Join EB, EC, FB, FC.

Then because in the triangles EGC, EGB, $CG = BG$ and EG in common, and the included angles are right angles, therefore $EB = EC$,
and therefore C lies on the given circle whose centre is E

In the same manner it may be proved that C lies on the given circle whose centre is B,
that is, the circles have another common point C

THEOREM 24

If two circles touch one another, the line through their centres passes through their point of contact.*

For if the point of contact were not in the line joining their centres, then the circles would have another common point, and therefore not touch one another

COR *Two circles that touch one another have a common tangent at the point of contact* [By Theor 18]

OBS (1) If the distance between the centres of two circles is greater than the sum of their radii, their circumferences will not meet and each circle will be wholly outside the other

OBS (2) If the distance between the centres of two circles is equal to the sum of their radii, their circumferences will meet in one point only, and each circle will lie outside the other

* Euclid, III 11, 12

Def. 14 In this case the circles are said to *touch externally.*

Obs (3) If the distance between the centres of two circles is less than the sum and greater than the difference of their radii, their circumferences will meet in two points

Def. 15. In this case the circles are said to *cut one another.*

Obs (4) If the distance between the centres of two circles is equal to the difference of their radii, their circumferences will meet in one point only, and one circle will lie within the other.

Def. 16 In this case the circles are said to *touch internally.*

Obs (5) If the distance between the centres of the two circles is less than the difference of their radii, their circumferences will not meet and one circle will be wholly within the other.

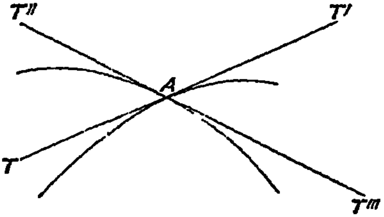
Obs (6). The converse of each of the above five Theorems is true [Rule of Conversion]

Treatment by Limits

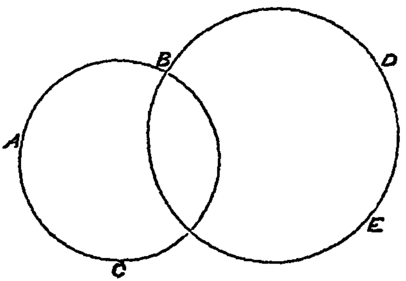
The results of Th 22 may also be readily obtained from Th 21 by means of the definition of Tangents given in Sect. IV. B For if the two common points, which form the extremities of the common chord spoken of in Th. 21, move up to coincidence, the common chord becomes a common tangent, and the line joining the centres must be

perpendicular to the common tangent, and pass through its point of contact In a similar way we may obtain Th. 23. For circles cannot meet in more than two points, and hence if these points move up to coincidence (so that the circles touch at that point) the circles can meet in no other point, and hence one must be wholly within the other or each must be wholly without the other

We can also demonstrate by the method of limits an important proposition converse to Th 23, viz. If two circles have but one common point they touch at that point For it immediately follows from the definition of a tangent given in Sect IV. B, that if two curves have a common point at which the tangents to the said curves make an angle with one another the curves must cross at that point



But it is evident that if the circumferences of two circles *ABC* and *DBE* cross at any point *B*, the circles must have another point common, for on one side of *B* the circumference of the circle *ABC* falls within the



other circle, and on the other side of *B* it is without the same, but circles are continuous curves, therefore the circumference of *ABC* must cross that of *BED* at some point

other than B. Hence if two circles have but one common point B they cannot cross one another there; and therefore their tangents at B cannot include an angle but must coincide

Exercises

1. If a straight line touch the inner of two concentric circles, and be terminated by the outer, prove that it will be bisected at the point of contact.

2. Any two chords which intersect on a diameter and make equal angles with it are equal.

3. Two fixed circles touch each other externally, and a third circle is described touching both externally. Shew that the difference of the distances of its centre from the centre of the two given circles will be constant.

4. If two circles intersect one another, and circles are drawn to touch both, prove that either the sum or the difference of the distances of their centres from the centres of the fixed circles will be constant, according as they touch (1) one internally and one externally, (2) both internally or both externally.

5. If two circles touch one another, any line through the point of contact will cut off segments from the two circles which contain the same angle.

6. If two circles touch one another, any two straight lines through the point of contact will cut off arcs, the chords of which are parallel

7 Two circles cut one another, and lines are drawn through the points of section and terminated by the circumference, shew that they intercept arcs the chords of which are parallel

8 Circles whose radii are 6·7 and 7·8 inches are successively placed so as to have their centres 14, 14½, and 15 inches apart. Shew whether the circles will meet or touch or not meet one another.

9 What will be the case if the centres are 1 inch, 1·1 inch, or 1·2 inches apart?

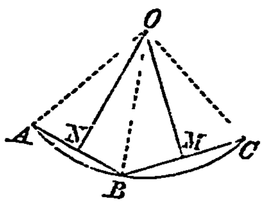
SECTION VI.

Problems.

Problem 1

*To find the centre of a given circle, or of a given arc**

Let ABC be the arc.



Construction Draw any two chords AB, BC, and bisect them at right angles (Book 1 Problems 2, 4) by straight lines ON, OM, which will intersect at O. O shall be the centre required

Proof. For NO is by construction the locus of points equidistant from A and B, and therefore $AO=BO$.

* Eucl III. 1.

Similarly, MO is the locus of points equidistant from B and C, therefore O is equidistant from A, B and C

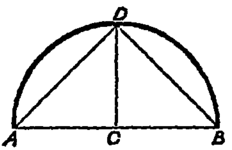
Hence, the circle described with centre O and radius equal to one of these three lines, will pass through the other two, and having three points coinciding with the given circular arc, must coincide with it throughout

(III 10 Cor. 1.)

PROBLEM 2.

*To bisect a given arc**

Let AB be the given arc; it is required to bisect it



Construction Join AB, and bisect AB at C, and draw CD at right angles to AB, to meet the arc in D

Then the arc AB is bisected in D

Proof Join AD, BD Then, since by construction, CD is the locus of points equidistant from A and B, therefore $AD = BD$

But equal chords cut off equal arcs, (III 5)

and therefore the arcs AD, BD are equal

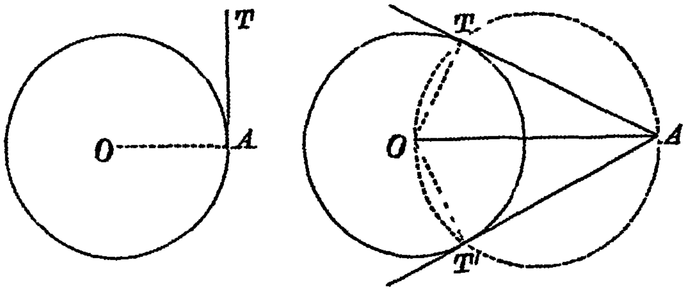
* Euclid, III 30

PROBLEM 3.

To draw a tangent to a circle from a given point on or outside the circumference.*

There will be two cases.

First, let the given point A be on the circumference. Let O be the centre



Construction Join OA, and draw AT at right angles to OA (I. Prob. 2).

Proof Then AT is a tangent. (Th. 18.)

Secondly, let A be outside the circle.

Construction. Join OA, and on it as diameter describe a circle, cutting the given circle in T and T'. Join AT, AT'; these shall be tangents from A.

Proof. Join OT, OT'. Then since ATO is a semicircle, the angle ATO is a right angle (III. 15). That is, AT or AT' is at right angles to the radius to the point where it meets the circumference, and therefore AT and AT' are tangents. (Th. 18.)

* Euclid, III. 17.

PROBLEM 4

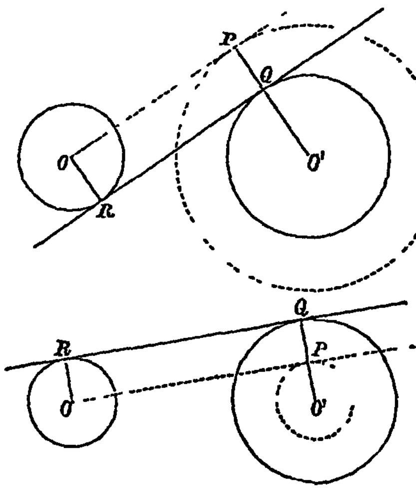
To draw a common tangent to two given circles.

Let the centres of the circles be O, O'

Construction With centre O' and radius equal to the sum or difference of the radii of the given circles, describe a circle, as in the figures

From O draw a tangent to this circle, touching it in P (III. Prob 3) Join $O'P$, and let it, produced through P if necessary, meet the circumference of the circle whose centre is O' in the point Q Through O draw OR parallel to PQ on the same side of OP as Q to meet the circle whose centre is O in R, and join QR QR will be a tangent to both circles

Proof Since PQ is by construction equal and parallel to OR, therefore RQ is parallel to OP. (I 30)



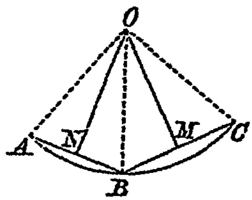
But OP is at right angles to $O'P$, since it touches the circle in P, and therefore RQ is at right angles to OQ, and it is also at right angles to OR, therefore it touches both circles

Cor. *When the circles are wholly outside one another, they have four common tangents: when they touch externally, they have three common tangents · when they intersect one another, they have two common tangents: when they touch internally, they have one common tangent: and when one of the circles is wholly inside the other, they have no common tangent.*

Problem 5

To describe a circle passing through three points which are not in the same straight line.

Let A, B, C be the three points which are not in the same straight line It is required to describe a circle to pass through A, B and C.



Construction. Join AB, BC. Bisect AB at right angles by the straight line NO, and bisect BC at right angles by the straight line MO, meeting the former in O Then with

centre O, at the distance OA, describe a circle. It will pass through B and C

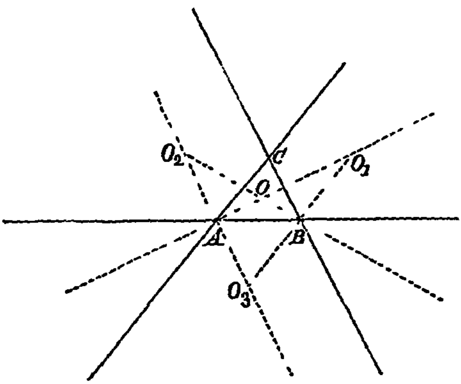
Proof Because, by construction, NO is the locus of points equidistant from A and B, therefore $OA = OB$ And because MO is the locus of points equidistant from B and C, therefore $BO = CO$ Therefore the circle described with centre O, and radius OA, will pass through A, B and C.

PROBLEM 6

To describe a circle to touch three given straight lines of indefinite length, which are not all parallel, and do not all pass through the same point.

Let the three given lines intersect in A, B, and C

Then, since the circle required is to touch the lines that intersect in A, its centre must lie on one of the bisectors



of the angles at A (III 20 Cor) Similarly, it must lie on

one of the bisectors of the angles at B. Therefore the construction is as follows:

Construction. Draw the bisectors of the angles at A and B, which will intersect in four points O, O_1, O_2, O_3.

These will be the centres of the circles required, and a circle described with any one of these points as centre, to touch one of the given lines, will touch the other two

COR. 1. *It follows that* COO_3 *and* O_2CO_1 *are straight lines, that is, the six bisectors of the interior and exterior angles of a triangle intersect one another three and three in four points.*

COR. 2. *If two of the lines are parallel, only two circles can be described to touch the three lines.*

COR 3. *If all the lines are parallel, or if they all pass through one point, no circle can be described to touch them all.*

Def. 17. A circle that touches the three sides of a triangle is called an *inscribed* circle.

Def. 18. A circle that touches one side of a triangle and the other two sides produced is called an *escribed* circle.

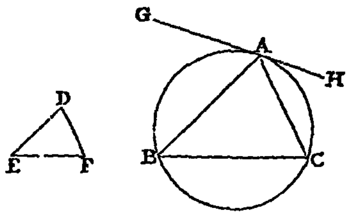
PROBLEM 7.

In a given circle to inscribe a triangle equiangular to a given triangle.

Construction. Let ABC be the given circle, and DEF the given triangle: it is required to inscribe in the circle ABC a triangle equiangular to the triangle DEF.

Draw the straight line GAH touching the circle at the point A;

at the point A, in the straight line AH, make the angle HAC equal to the angle DEF,



and, at the point A, in the straight line AG, make the angle GAB equal to the angle DFE, and join BC

ABC shall be the triangle required.

Proof. Because GAH touches the circle ABC, and AC is drawn from the point of contact A,

therefore the angle HAC is equal to the angle ABC in the alternate segment of the circle (III 19)

But the angle HAC is equal to the angle DEF

Therefore the angle ABC is equal to the angle DEF.

For the same reason the angle ACB is equal to the angle DFE.

Therefore the remaining angle BAC is equal to the remaining angle EDF.

Wherefore the triangle ABC is equiangular to the triangle DEF, and it is inscribed in the circle ABC*.

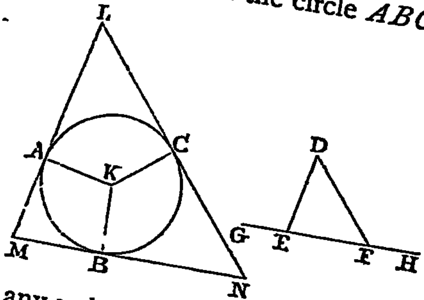
PROBLEM 8

About a given circle to circumscribe a triangle equiangular to a given triangle.

* Euclid, IV. 2.

Let ABC be the given circle, and DEF the given triangle: it is required to describe a triangle about the circle ABC, equiangular to the triangle DEF.

Construction. Produce EF both ways to the points G, H, take K the centre of the circle ABC;



from K draw any radius KB;

at the point K, in the straight line KB, make the angle BKA equal to the angle DEG, and the angle BKC equal to the angle DFH,

and through the points A, B, C, draw the straight lines LAM, MBN, NCL, touching the circle ABC.

LMN shall be the triangle required.

Proof. Because LM, MN, NL touch the circle ABC at the points A, B, C,

to which from the centre are drawn KA, KB, KC,

therefore the angles at the points A, B, C are right angles

And because the four angles of the quadrilateral figure $AMBK$ are together equal to four right angles,

(for it can be divided into two triangles,)

and that two of them KAM, KBM are right angles,

therefore the other two AKB, AMB are together equal to two right angles

But the angles DEG, DEF are together equal to two right angles.

Therefore the angles AKB, AMB are equal to the angles DEG, DEF,

of which the angle AKB is equal to the angle DEG,

therefore the remaining angle AMB is equal to the remaining angle DEF.

In the same manner the angle LMN may be shewn to be equal to the angle DFE

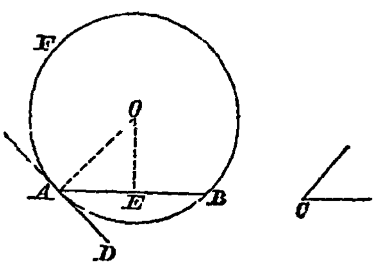
Therefore the remaining angle MLN is equal to the remaining angle EDF.

Wherefore the triangle LMN is equiangular to the triangle DEF, and it is described about the circle ABC*.

PROBLEM 9.

On a given straight line to describe a segment of a circle containing an angle equal to a given angle.

Let AB be the given line, C the given angle.



* Euclid, IV 3.

Construction. At the point A make an angle BAD equal to the angle C (I. Prob. 6).

Then if a circle be described to touch AD in A, and to pass through B, the segment of that circle alternate to BAD will be the segment required. (Th 19)

To find the centre of this circle, draw AO at right angles to AD: then AO is the locus of the centres of all circles which touch AD at A. (III 18 c.)

Bisect AB at right angles by the line EO; then EO is the locus of the centres of circles which pass through A and B. (III 8. Cor)

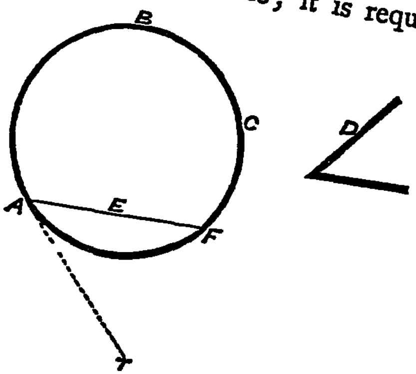
Therefore O, the point of intersection of these lines, is the centre of the circle required.

With centre O and radius OA or OB describe a circle, which will touch AD at A and pass through B, and therefore the segment AFB contains an angle equal to the alternate angle BAD, that is to the given angle C*.

PROBLEM 10

From a given circle to cut off a segment containing a given angle.

Let ABC be the given circle; it is required to cut off



* Euclid, III 23.

from it a segment containing an angle equal to the given angle D

Construction At A any point on the circumference draw the tangent AT, (III Prob 3)

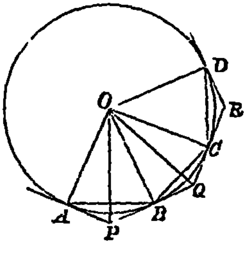
and at the point A in the straight line AT make the angle TAE equal to D (I. Prob 6), and let AE meet the circle again in F,

then the segment cut off by AF, remote from AT, shall contain an angle equal to the alternate angle FAT (III 19), and therefore the segment ABF contains an angle equal to D*.

THEOREM 25

If the whole circumference of a circle is divided into any number of equal arcs, the inscribed polygon formed by the chords of these arcs is regular, and the circumscribed polygon formed by tangents drawn at all the points of division is also regular.

Part. En Let the circumference of the circle ABC be



* Euclid, III. 34.

divided into any number of equal arcs in the points A, B, C, D.... it is required to prove that the polygon $ABCD$... is regular, and that so is also the polygon formed by tangents drawn at the points A, B, C ...

Proof. Because the minor arcs AB, BC, CD .. are all equal, the chords AB, BC, CD .. are equal,

and therefore the polygon $ABCD$... is equilateral. Also each of the angles ABC, BCD . stands upon an arc that is made up of all but two of the equal arcs into which the circle is divided; thus they stand upon equal arcs, and are therefore equal, and therefore the polygon is also equiangular.

It is therefore regular.

Again, draw tangents at the points A, B, C. . and let them form the polygon PQR.... Take O the centre of the circle and join OA, OB.

Because the interior angles of the quadrilateral $OAPB$ are equal to four right angles (I. 26.)

and those at A and B are right angles:

therefore the angles APB and AOB are together equal to two right angles, and APB is the supplement of AOB.

Similarly, each of the other angles of the circumscribing polygon is supplemental to one of the angles at the centre that stand upon the equal arcs into which the circle has been divided, and which are therefore equal.

Hence the polygon is equiangular.

Again, join OP, OQ. Then because the tangents from P make equal angles with the line PO to the centre of the circle, the angle OPB is one half the angle APB.

Similarly OQB is half the angle BQC, which has been shewn to be equal to APB

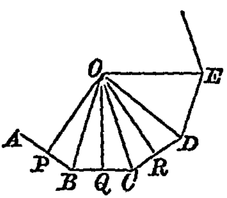
Therefore the angle OPB is equal to the angle OQB; and therefore $OP = OQ$ (I 8)

Similarly it can be shewn that $OQ = OR$, and thus all the lines from O to the angular points of the circumscribing circle are equal, and a circle may be described with centre O passing through them all Describe it, then since OA, OB, and OC are all equal therefore the sides PQ, QR, of the polygon are chords in it equally distant from the centre and are therefore equal Hence the polygon formed by the tangents at A, B, C,. is equilateral, and is therefore a regular polygon

THEOREM 26

If straight lines are drawn bisecting two angles of a regular polygon, the point in which the bisectors intersect is equidistant from all the vertices of the polygon and from all the sides

Part En Let $ABCDE$ be a regular polygon, and let BO, CO be drawn bisecting the adjoining angles ABC



and BCD. It is required to prove that the point O in which they meet is equidistant from all the angles and all

the sides of the polygon, and that it lies on all other bisectors of the angles of the polygon

Proof. Join OD. Then because in the triangles OBC, ODC

$$BC = CD \qquad \text{(Hyp.)}$$

$$CO \text{ is common,}$$

$$\text{and } BCO = DCO; \qquad \text{(Hyp.)}$$

therefore the triangles are equal, and

the angle $CBO = CDO$;

and therefore CDO is equal to one half one of the angles of the regular polygon. Therefore the line OD bisects the angle CDE.

And similarly we may shew that the line from O to each angular point of the polygon bisects that angle of the polygon.

Again, because OCD and ODC are each the half of an angle of the polygon, they are equal, and the side $OC = OD$.

Similarly each of the lines OA, OB, and OC is equal to the next,

and therefore they are all equal,

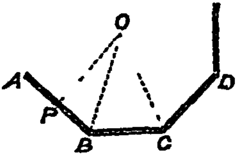
and a circle can be described with centre O passing through all the angular points of the polygon. Moreover the sides are equal chords in this circle, and are therefore equally distant from the centre O.

Hence O lies on the bisector of each angle of the polygon, and is equidistant from all its sides and angular points.

PROBLEM 11.

To inscribe a circle in, or to circumscribe a circle about, a regular figure

Let $ABCD$ be a regular figure, it is required to in-



scribe a circle in it, and also to circumscribe a circle about it.

Construction. Bisect the two adjacent angles ABC, BCD of the figure by the lines BO, CO meeting in O From O draw the perpendicular OP on AB With centre O and radius OP describe a circle, it shall be inscribed in the figure $ABCD$, and with centre O and radius OB describe a circle, it shall be circumscribed about the said figure.

Proof Because BO and CO bisect two adjacent angles of the regular figure $ABCD$ the point O where they meet is equidistant from all the sides of that figure (Th 26)

Therefore the circle whose centre is O and radius OP will pass through the feet of all the perpendiculars from O upon the sides of the figure, and the said sides, being perpendicular to the radii drawn to the points where they meet the circle, will touch the circle, therefore the circle is inscribed in the regular figure $ABCD$. . .

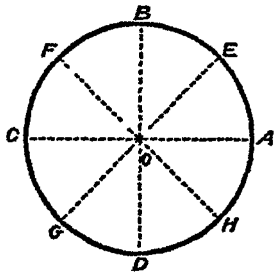
Similarly, because O is equidistant from all the angles of the polygon, (III 26)

therefore the circle described with centre O and radius OB will be circumscribed to the polygon.

PROBLEM 12.

To inscribe in, or to circumscribe about, a given circle regular figures of 4, 8, 16, 32,... *sides.*

Construction. Let O be the centre of the circle. Draw two diameters AOC, BOD at right angles to one another



Then because the four angles that they form at the centre are equal, the points A, B, C, D divide the circumference of the circle into four equal arcs.

Again, by bisecting each of the angles thus formed at the centre by lines meeting the circumference, we shall divide the circumference into 8 equal arcs, and by repeating the process we can divide it into 16, 32.. equal parts.

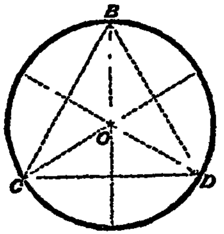
Then the chords of the equal arcs will form a regular inscribed figure of the prescribed number of sides, (III. 25)

and the tangents at the points of division will form a regular circumscribed figure of the prescribed number of sides. (III. 25)

PROBLEM 13

To inscribe in, or to circumscribe about, a given circle regular figures of 3, 6, 12, 24 *sides.*

Construction. Let O be the centre of the circle. Inscribe in the circle the equilateral triangle, BCD (III.



Prob 7), and join OB, OC, OD Then the angles BOC, COD, DOA are equal, (III 5 and 3)

and by bisecting them by lines meeting the circumference we shall divide the circumference into six equal arcs Again, by bisecting the angles which the said arcs subtend at the centre we shall divide the circumference into 12 equal arcs, and by repeating the process we shall divide it into 24, 48 equal parts.

Then the chords of the equal arcs will form a regular inscribed figure of the prescribed number of sides (III 25)

And the tangents at the points of division will form a regular circumscribed figure of the prescribed number of sides. (III 25)

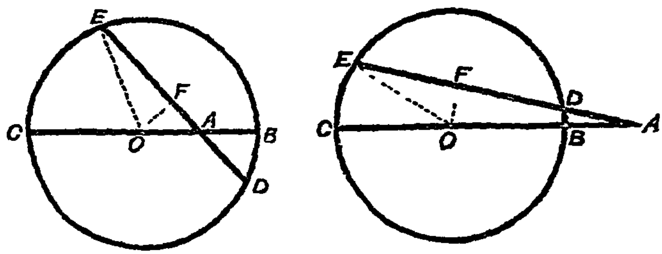
SECTION VII

The Circle in connection with Areas.

Theorem 27.

If a chord of a circle is divided into two segments by a point in the chord or in the chord produced, the rectangle contained by these segments is equal to the difference of the squares on the radius and on the line joining the given point with the centre of the circle.

Part En. Let A be a point and CEB a circle, whose centre is O. Then the rectangle contained by the segments into which A divides any chord through it, shall be equal to the difference between the squares on OA and on the radius.



Proof Join AO and let it cut the circle in B and C, and draw through A any other chord DE not at right angles to AO Draw OF the perpendicular from O upon it and join OE

Then because the square on AO is equal to the squares on OF, FA, (II 9)

and the square on OE is equal to the squares on OF, FE,

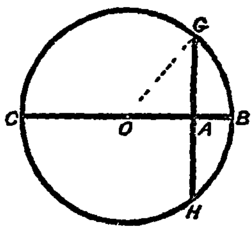
therefore the difference between the squares on OA and OE is equal to the difference between the squares on AF, FE,

that is, to the rectangle contained by the sum and difference of AF and FE. (II 8)

But AE is the sum of AF, FE, and AD is the difference between AF, FE, since $FE = FD$; (III 7)

therefore the rectangle contained by AE, AD is equal to the difference of the squares on AO and the radius

Again, if A be within the circle and GAH be the chord bisected at A and OG be joined, it is obvious that because GAO is a right angle the difference between the squares of OA and the radius is equal to the square of AG, that is, to the rectangle under GA, AH



Therefore if any chord be drawn through A the rectangle under the segments into which it is divided internally or externally by A is equal to the difference between the squares of the radius and the distance of A from the centre of the circle

Cor. 1. *The rectangle contained by the segments of any chord of a circle passing through a given point is the same, whatever be the direction of the chord*.*

Cor. 2. *If the point is within the circle, the rectangle contained by the segments of any chord passing through it is equal to the square on half that chord which is bisected by the given point*

Cor 3. *If the point is without the circle, the rectangle contained by the segments of any chord passing through it is equal to the square on the tangent to the circle drawn from that point†.*

For if OT be the tangent its square is equal to the difference between the squares of OA and the radius, since the angle OTA is a right angle.

Cor. 4. *Conversely. if the rectangle contained by the segments of a chord passing through an external point is equal to the square of a line joining that point with a point in the circumference of the circle, this line touches the circle§*

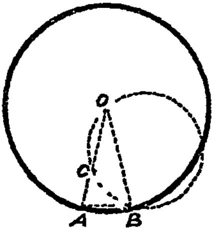
For by the last corollary it must be equal in length to each of the two tangents from the point, and therefore must be one of them, since by Theorem 10, Cor. 3, there cannot be more than two equal straight lines drawn to the circle from a point not the centre

* Euclid, III 35 † Euclid, III. 36 § Euclid, III 37

PROBLEM 14.

To inscribe in a circle a regular decagon, and thence to circumscribe a regular decagon about a circle, also to inscribe in, or to circumscribe about, a given circle a regular pentagon, or regular figures of 20, 40, 80 *sides*

Construction Let O be the centre of the circle Take any radius OA and divide it in C so that the rectangle



under OA and AC is equal to the square on OC.

(II Prob 5)

Draw a chord AB of the circle equal to OC It shall cut off an arc equal to one-tenth part of the whole circumference

Proof Join OB and CB, and describe a circle round the triangle OBC Then because AB is equal to OC, the rectangle under OA and AC is equal to the square of AB (Constr)

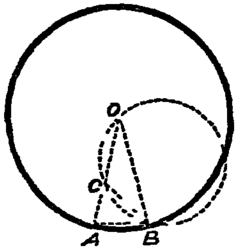
And because the rectangle under OA, AC the segments of a chord of the circle OBC drawn from the external point A, is equal to the square of the line joining A with a point B on that circle, therefore AB touches that circle,

(III 27, Cor 4)

and BC is a chord drawn from the point of contact,

therefore the angle CBA is equal to the angle BOC in the alternate segment (III 19)

and the angle BAC is common to the two triangles BCA, AOB,



therefore the third angle ACB of the one is equal to the third angle ABO of the other:

but the angle ABO is equal to the angle BAO, because OA is equal to OB; (I 6)

therefore the angle ACB is equal to the angle BAC, and therefore the side BC is equal to the side AB, and therefore to CO; (Constr)

therefore OBC is an isosceles triangle, and the exterior angle ACB at the vertex is double of the angle BOC, one of the equal angles at the base, (I 25)

but OAB and OBA are each of them equal to ACB, therefore they are each double the angle AOB, which is the remaining angle of the triangle AOB,

therefore the angle AOB is one-fifth part of the sum of the angles of the triangle AOB, that is of two right angles,

therefore it is one-tenth part of four right angles,

therefore the arc AB on which it stands is one-tenth part of the whole circumference.

We can thus divide the circumference into ten equal parts, and so into 20, 40, &c. by bisecting the said equal

parts or the angles they subtend at the centre, also by taking every alternate point of division we can divide the circumference into five equal parts,

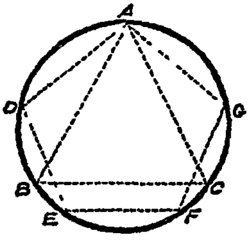
and thus by joining the points of division, or drawing tangents to the circle thereat, we can inscribe and circumscribe about the circle regular figures of 5, 10, 20.. sides*

(III. 25)

PROBLEM 15

To inscribe in a circle a regular quindecagon, and thence to circumscribe a regular quindecagon about a circle, also to inscribe in, or to circumscribe about, a given circle regular figures of 30, 60, 120 sides

Construction Let AD, DE be sides of a regular pentagon inscribed in the circle, and AB the side of an equilateral



triangle inscribed in the circle Then BE shall be the fifteenth part of the circumference

Proof. Because AE is two-fifths of the circumference and AB is one-third, therefore BE is one-fifteenth part of the circumference,

and by proceeding as in Problems 12, 13, and 14 we can circumscribe a regular quindecagon about a circle, and also inscribe in, or circumscribe about, a given circle regular figures of 30, 60, 120 sides†.

* Euclid, IV. 10. † Euclid, IV 16

Remark. Regular polygons can therefore be constructed when the number of their sides is 3, 4, 5, or 15, or these numbers multiplied by any power of 2 And besides these no other regular polygons can be constructed by the use of the ruler and compasses only, with the remarkable exception discovered by Gauss; who shewed that a polygon of 2^n+1 sides can be described by the ruler and compasses alone, when n is such that 2^n+1 is a prime number. If n has the values 1, 2, 3 . in succession, 2^n+1 takes the values 3, 5, 9, 17, 33, 65, 129, 257 of which 3, 5, 17, 257 are primes. Hence Gauss has shewn that regular polygons of 17 and 257 sides *can* be constructed by the use of the ruler and compasses, but the construction and proof, even for the first of these, are far too tedious to be given in an elementary work

EXAMPLES ON BOOK III.

1. Two circles touch one another in A, and have a common tangent BC. Shew that the angle BAC is a right angle.

2. AOB, COD are two chords of a circle at right angles to one another, prove that the squares of OA, OB, OC, and OD, are together equal to the square of the diameter

3. With the same hypothesis, if M is the centre, prove that $AB^2+CD^2=8AM^2-4OM^2$.

4. Describe a circle to touch a given line in a given point, and pass through another given point.

5 Describe a circle to touch a given circle in a given point, and to pass through another given point.

6. Find the locus of the point of intersection of the lines which bisect the angles at the base of triangles on the same base and having a given vertical angle.

(Prove that the angle between each pair of bisectors is the same.)

7 Find the locus of the points of bisection of equal chords in a circle.

8. Find the locus of the centres of circles which touch a given circle in a given point.

9 Find the locus of the middle point of a line drawn from a given point to meet a given circle.

10 Shew that the inscribed equilateral triangle is one fourth of the circumscribed equilateral triangle.

11 A ladder slips down a wall find the locus of its middle point

12 If from two fixed points in the circumference of a circle two lines are drawn to intercept a given arc, the locus of their intersections is a circle

13 Two chords of a circle which do not bisect each other do not both pass through the centre

14. Two shillings are moved in the corner of a box so that each always touches one side, and they touch one another, find the locus of the point of contact

15. Two circles cut one another, and lines are drawn through the points of section, and terminated by the circumferences; shew that the chords which join the extremities of these lines are parallel

16. Two equal circles intersect in A and B, and any line BCD is drawn to cut both circles Prove that

$$AC = AD$$

17. Two equal circles intersect in A, B, a third circle is drawn, with centre A and any radius less than AB, meeting the circles in C, D, on the same side of AB. Prove that B, C, D lie in one straight line

18. ACD, ADB are two segments of circles on the same base AB; take any point C on the segment ACB, and join CA, CB, and produce them if necessary to meet ADB in D, E. Shew that the arc DE is constant.

19 If two circles cut each other, and from either point of intersection diameters be drawn, the extremities of these diameters and the other point of intersection shall be in the same straight line

20. If a straight line that touches a circle be parallel to a chord of it, the point of contact will bisect the arc cut off by that chord.

21 Perpendiculars AD, CE are let fall from the angles A, C of the triangle ABC on the opposite sides Prove that the angle ACE is equal to the angle ADE.

22 Two circles intersect in A, B, and tangents AC, AD are drawn to each circle, meeting circumferences in C, D, prove that BC, BD make equal angles with BA.

23 If one of two intersecting circles pass through the centre of the other, prove that the tangent to the first at the point of intersection, and the common chord, make equal angles with the radius to that point from the centre of the second.

24. Given base, altitude, and vertical angle, construct the triangle.

25 To draw a line from a given point such that the perpendicular on it from a given point shall have a given length.

26. In a given straight line to find a point at which a given straight line subtends a given angle.

27. Describe a circle to touch a given circle, and touch a given line in a given point

28 Describe a circle of given radius to touch a given line, and have its centre on another given line.

29 A given chord of a circle is produced Find a point in it from which the tangents to the circle shall have a given length.

30. With a given radius describe a circle touching two given circles.

31 Describe a triangle, having given the vertical angle and the segments of the base made by the line bisecting the vertical angle.

32 Given base, altitude, and radius of circumscribed circle, construct the triangle

33 The triangle contained by the two tangents to a circle from any point and any other tangent that meets them and lies between the point and the circle has its perimeter double of either of the two tangents Prove this, and apply it to construct a triangle, having given the vertical angle, perimeter, and altitude.

34 Given the perimeter, the vertical angle, and the line bisecting the vertical angle, construct the triangle.

35 The chord AB is produced both ways equally to C, D, and tangents CE, DF are drawn on opposite sides of CD, shew that EF bisects AB.

36 The three perpendiculars to the sides of a triangle drawn through their middle points meet in one point.

37. The three lines which join the angles of a triangle to the middle points of the opposite sides intersect in one point.

38. The three perpendiculars from the angles of a triangle on its opposite sides intersect in one point.

39 If two circles touch one another, the lines which join the extremities of parallel diameters towards opposite parts will intersect in the point of contact.

40 The circles described on the sides of a triangle as diameters intersect in the sides, or sides produced, of the triangle.

41 Equilateral triangles are described externally on the sides of a triangle; prove that the circles described about those triangles pass through one point.

42. The four common tangents to two circles which do not meet one another intersect, two and two, on the straight line which joins the centres of the circles

43 Given the altitude, the bisector of the vertical angle, and the bisector of the base, to construct the triangle.

44. The three circles which pass through two angles of a triangle and the point of intersection of the perpendiculars of the triangle are each equal to the circumscribing circle.

45 If a triangle is equilateral, shew that the radii of the inscribed, the circumscribed, and an escribed circle are to one another as 1, 2, 3

46 If circles are described with the vertices of a triangle as centres, and so as to pass through the points of contact of the inscribed circle with the adjacent sides, these three circles will touch one another.

47 Place a straight line of given length in a circle so that it shall be parallel to a given diameter of the circle.

48 Place (when possible) a straight line of given length in a circle so that it shall pass through a given point within or without the circle.

49 Given three points, describe circles from them as centres so that each may touch the other two

50 On the side of any triangle equilateral triangles are described externally, and their vertices joined to the opposite vertices of the given triangle, shew that the joining lines pass through one point.

51. O is the centre of the circle inscribed in the triangle ABC, which touches AB, AC in C', B'; if AO cuts the circle in P, and AO produced in P', shew that P, P' are the centres of the inscribed and escribed circles of the triangle $AB'C'$.

52. Shew that the area of a triangle is equal to the rectangle contained by its semi-perimeter and the radius of the inscribed circle.

53 Of all the rectangles inscribable in a circle, shew that a square is the greatest

54 Can a circle be inscribed in (1) a rectangle, (2) a parallelogram, (3) a rhombus?

55. Shew that the inscribed hexagon is three-fourths of the circumscribed hexagon.

56. To find four points such that the line joining every two may be perpendicular to the line joining the other two.

57 Shew that the six segments into which the points of contact of the escribed circles of a triangle divide the sides, may be arranged in three pairs of equal segments.

58. Inscribe an octagon in a given circle.

59. Describe a circle to intercept equal chords of any given length on three given straight lines

In how many ways may this problem be solved?

60. At any point in the circumference of the circle circumscribing a square, shew that one of the sides subtends an angle three times as great as the others.

61. Find the locus of points at which two sides of a square subtend equal angles.

62. Find the locus of points at which three sides of a square subtend equal angles.

63. If four straight lines intersect one another so as to form four triangles, prove that the four circumscribing circles will pass through one point.

64. Of all triangles that can be inscribed in a circle the greatest is the equilateral. Extend this to the case of a polygon of any number of sides

65. Of all triangles that can be inscribed in a given triangle that whose angles are the feet of the perpendiculars of the original triangle has the smallest perimeter

66. A straight line is divided into any two parts in C, and ADC, CEB are equilateral triangles on the same side of AB. Find the locus of the intersection of AE and BD.

67. If from any point on the circle circumscribing a triangle perpendiculars be let fall upon the sides, the feet of these perpendiculars lie in one straight line.

PROBLEM PAPERS ON THE TRIANGLE AND ITS ASSOCIATED CIRCLES.

No I

1 The three bisectors of the angles of a triangle pass through one point (O), and this is the centre of the inscribed circle

2. The three straight lines which bisect one angle of a triangle and the other two exterior angles meet in one point (O'), and this is the centre of an escribed circle of the triangle

3. If the inscribed circle of the triangle ABC touches the sides opposite A, B, C in Q, R, S, and the circle escribed to the side opposite A, touches the sides or sides produced opposite to A, B, C, in X, Y, Z, prove that

$$AZ = AY = \tfrac{1}{2} \text{ perimeter of triangle,}$$
$$\text{and } QX = AB \sim AC,$$
$$\text{and } CR = BZ.$$

4 In the same figure prove that if a, b, c are the lengths of the sides, s half their sum, $AR = s - a$

No II

5 Prove that the three perpendiculars drawn to the sides of a triangle through their middle points meet in one point, which is the centre of the circumscribing circle.

6 Prove that the three perpendiculars drawn to the sides of a triangle from the opposite angles intersect in one point. [This point is often called the orthocentre]

{This may be deduced from (5), by drawing through each vertex a parallel to the opposite side (Catalan)}

7. Prove that the three medians of a triangle intersect in one point (called the centre of gravity), which is a point of trisection of each median.

8. If G is the centre of gravity of the triangle ABC, prove that the triangles GAB, GBC, GCA are all equal.

No III

9 If G is the centre of gravity of the triangle ABC, prove that

$$GA^2 + GB^2 + GC^2 = \frac{AB^2 + BC^2 + CA^2}{3}.$$

10. The centre of the circumscribing circle (I), the centre of gravity (G), and the point of intersection of the perpendiculars (P), lie in one straight line, and

$$IG = \tfrac{1}{2} GP.$$

{If F is the middle point of AC and E of AB, prove the triangles EIF, CPB similar, and $IF = \frac{1}{2} BP$.}

11. The circles which pass through two vertices of a triangle, and the intersection of the perpendiculars, will be equal to the circumscribing circle.

12. The angles BIC, CIA, AIC are respectively double of the angles at A, B, C.

No. IV.

13. If Q, R, S, are the feet of the perpendiculars let fall from A, B, C on the opposite sides, prove that AQ, BR, CS, are the bisectors of the angles of the triangle QRS.

14. Prove that if I is the centre of the circumscribing circle, and Q, R, S as in (13), IA, IB, IC are respectively perpendicular to RS, SQ, QR.

15. From a centre O describe a circle; from a point G on its circumference describe a second circle cutting the former in B, C and from a point I on the second circle describe a circle to touch BC Prove that the other tangents from B, C to the third circle will intersect on the circumference of the first.

16. Hence shew that if I, I' are centres of the inscribed and escribed circles of a triangle, II' is bisected by the circumference of the circumscribing circle.

No V

17. I is the centre of the circumscribing circle, P the intersection of the perpendiculars; E, F, G the middle points of BC, CA, AB; Q, R, S the feet of the perpendiculars from A, B, C on those sides H the middle point of IP. Prove that H is the centre of a circle which bisects PA, PB, PC, and that its radius is half that of the circumscribing circle.

18. If L, M, N are the middle points of PA, PB, PC, prove that $IE = PL$, and hence that EL is bisected in H.

19 Hence prove that the circle LMN also passes through E, F, G, and through Q, R, S

[This circle is therefore called the nine point circle]

NOTE. The advanced student will do well to get Catalan's *Théorêmes et Problêmes de Géométrie Elémentaire.*

BOOK IV.

Fundamental Propositions of Proportion.

SECTION I.

Of Ratio and Proportion.

[Although a complete treatment of Proportion, such as that contained in this Book, is indispensable to a sound knowledge of Geometry, Book V. may be read immediately after Book III by students who are acquainted with the treatment of Ratio and Proportion given in books on Arithmetic and Algebra.]

[*Notation*

In what follows, large Roman letters, A, B, etc , are used to denote magnitudes, and where the pairs of magnitudes compared are both of the same kind they are denoted by letters taken from the early part of the alphabet, as A, B compared with C, D; but where they are or may be of different kinds, from different parts of the alphabet, as A, B compared with P, Q or X, Y. Small Italic letters m, n, etc., denote whole numbers By $m\,.\,$A or mA is denoted the mth multiple of A, and it may be read as m times A. The product of the numbers m and n is denoted by mn, and it is assumed that $mn=nm$. The combination $m\,.\,n$A denotes the mth multiple of the nth multiple of A and may be read as m times nA, and mnA or $mn\,.\,$A as mn times A. By $(m+n)$ A is denoted $m+n$ times A.]

Def 1. One magnitude is said to be a *multiple* of another magnitude when the former contains the latter an exact number of times.

According as the number of times is 1, 2, 3 m, so is the multiple said to be the 1st, 2nd, 3rd, . mth.

Def. 2 One magnitude is said to be a *measure* or *part* of another magnitude when the former is contained an exact number of times in the latter

The following property of multiples is axiomatic —

1. As $A >=$ or $< B$, so is $mA >=$ or $< mB$ (*Euc Ax* 1 & 3)

The converse necessarily follows, so that

2 As $mA >=$ or $< mB$, so is $A >=$ or $< B$ (*Euc Ax* 2 & 4)

The following theorems are easily proved —

3 $mA+mB+ \quad = m(A+B+ \ . \ )$ (*Euc.* v 1)

4 $mA-mB=m(A-B)$ (A being greater than B) (*Euc* v. 5)

5. $mA+nA=(m+n)A$ (*Euc* v 2)

6 $mA-nA=(m-n)A$ (m being greater than n) (*Euc* v 6)

7 $m.nA=mn.A=nm \ A=n.mA$ (*Euc* v 3).

Def 3. The *ratio* of one magnitude to another of the same kind is the relation of the former to the latter in respect of *quantuplicity.*

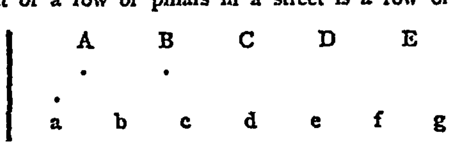
The ratio of A to B is denoted thus, A B, and A is called the *antecedent* of the ratio, B the *consequent*

The *quantuplicity* of A with respect to B may be estimated by examining how the multiples of A are distributed among the multiples of B, when both are arranged in ascending order of magnitude and the series of multiples continued without limit

OBS This interdistribution of multiples is *definite* for two given magnitudes A and B, and is different from that for A and C, if C differ from B by any magnitude however small See Th 4

This is a very important definition, and may be illustrated as follows (The illustration is due to De Morgan)

In front of a row of pillars in a street is a row of palings, the



pillars being A, B, &c, the palings a, b, c . in the figure.

Then there is a certain interdistribution of the pillars among the palings, or in other words of the multiples of the distance between the pillars among the multiples of the distance between the palings. The 2nd pillar lies between the 2nd and 3rd palings, the 5th pillar between the 6th and 7th palings, and so on

Now if the distance between the palings or between the pillars were altered by any quantity however small, then the distribution would be changed if the series were continued without limit For if the distance between the palings were changed by a distance equal to say the nth part of the distance between the pillars, then the nth paling would be changed by the whole distance between two pillars, and therefore its position among the pillars would be changed.

Def 4 The ratio of two magnitudes is said to be equal to that of two other magnitudes (whether of the same or of a different kind from the former), when any equimultiples whatever of the antecedents of the ratios being taken and likewise any equimultiples whatever of the consequents, the multiple of one antecedent is greater than, equal to, or less than that of its consequent, according as that of the other antecedent is greater than, equal to, or less than that of its consequent.

Or in other words:

The ratio of A to B is equal to that of P to Q, when mA is greater than, equal to, or less than nB, according as mP is greater than, equal to, or less than nQ, whatever whole numbers m and n may be.

It is an immediate consequence that:

The ratio of A to B is equal to that of P to Q; when, m being any number whatever, and n another number determined so that either mA is between nB and $(n+1)$ B or equal to nB, according as mA is between nB and $(n+1)$ B or is equal to nB, so is mP between nQ and $(n+1)$ Q or equal to nQ

The definition may also be expressed thus:

The ratio of A to B is equal to that of P to Q when the multiples of A are distributed among those of B in the same manner as the multiples of P are among those of Q.

That is, if a model were constructed of the pillars and palings, it would be correct, or the ratio of the distances of pillars and palings in the street is the same as the ratio of the distance of pillars and palings in the model, if every pillar in the model fell between the same palings in the model, as the corresponding pillar in the street did among the palings in the street, the street being supposed to be of indefinite length

It will be observed that this is a method of ascertaining whether four magnitudes are in proportion which is wholly independent of any arithmetical representation of the numbers

Def 5. The ratio of two magnitudes is greater than that of two other magnitudes, when equimultiples of the antecedents and equimultiples of the consequents can be found such that, while the multiple of the antecedent of the first is greater than or equal to that of its consequent, the multiple of the antecedent of the other is not greater or is less than that of its consequent.

Or in other words

The ratio of A to B is greater than that of P to Q, when whole numbers m and n can be found, such that, while mA is greater than nB, mP is not greater than nQ, or while $m\text{A}=n\text{B}$, mP is less than nQ.

Def 6 When the ratio of A to B is equal to that of P to Q, the four magnitudes are said to be *proportionals* or to form a *proportion*. The proportion is denoted thus.

$$\text{A} : \text{B} :: \text{P} : \text{Q},$$

which is read, "A is to B as P is to Q" A and Q are called the *extremes*, B and P the *means*, and Q is said to be the *fourth proportional* to A, B and P.

The antecedents A, P are said to be *homologous**, and so are the consequents, B, Q.

* That is, occupy the same position in the ratio.

Def. 7. Three magnitudes (A, B, C) of the same kind are said to be proportionals, when the ratio of the first to the second is equal to that of the second to the third: that is when A : B :: B : C.

In this case C is said to be the *third proportional* to A and B, and B the *mean proportional* between A and C.

Def. 8. The ratio of any magnitude to an equal magnitude is said to be a *ratio of equality.* If A be greater than B, the ratio A : B is said to be a *ratio of greater inequality*, and the ratio B : A a *ratio of less inequality.* Also the ratios A : B and B : A are said to be *reciprocal* to one another

THEOREM 1.

Ratios that are equal to the same ratio are equal to one another.

Proof. Let A : B :: P : Q, and also A : B :: X : Y, then shall P : Q :: X : Y.

For since $mA >=< nB$ according as $mP >=< nQ$ (Def. 4.)

and $mA >=< nB$ according as $mX >=< nY$,

therefore $mP >=< nQ$ according as $mX >=< nY$,

and therefore (Def 4) P : Q :: X : Y.

THEOREM 2.

If two ratios are equal, as the antecedent of the first is greater than, equal to, or less than its consequent, so is the antecedent of the second greater than, equal to, or less than its consequent.

Proof. Let A . B · P : Q, then as A >=< B so is P >=< Q.

For by Def 4, as mA >=< nB so mP >=< nQ, whatever integers m and n are. Let m and n each equal 1; then as A >=< B so P >=< Q

THEOREM 3

If two ratios are equal, their reciprocal ratios are equal.

Proof. Let A B P · Q, then B A Q P.

For, since the multiples of A are distributed among those of B as the multiples of P among those of Q, the multiples of B are distributed among those of A as the multiples of Q among those of P; and therefore

B : A : Q P. (Def. 4)

THEOREM 4.

If the ratios of each of two magnitudes to a third magnitude be taken, the first ratio will be greater than, equal to, or less than the other as the first magnitude is greater than, equal to, or less than the other and if the ratios of one magnitude to each of two others be taken, the first ratio will be greater than, equal to, or less than the other as the first of the two magnitudes is less than, equal to, or greater than the other.

Proof. Let A, B, C be three magnitudes of the same kind, then

A . C > = or < B . C, as A > = or < B,

and C A > = or < C . B, as A < = or > B

If A = B, it follows directly from Def 4 that A · C · B : C and C : A B : A.

If A > B, m can be found such that mB is less than mA by a greater magnitude than C.

Hence if mA be between nC and $(m+1)$C, or if $m\text{A}=n\text{C}$, mB will be less than nC, whence (Def. 6) A : C > B : C;

Also, since $n\text{C} > m\text{B}$ while nC is not $> m$A (Def. 6) C : B > C : A or C : A < C : B.

If A < B, then B > A and therefore B : C > A : C, that is A : C < B : C, and so also C : A > C : B.

COR. *The converses of both parts of the proposition are true, since the "Rule of Conversion" is applicable.*

THEOREM 5.

The ratio of equimultiples of two magnitudes is equal to that of the magnitudes themselves.

Proof. Let A, B be two magnitudes, then mA : mB :: A : B.

For as $p\text{A} >=$ or $< q\text{B}$, so is $m\,.\,p\text{A} >=$ or $< m\,.\,q\text{B}$; but $m\,.\,p\text{A} = p\,.\,m\text{A}$ and $m\,.\,q\text{B} = q\,.\,m\text{B}$, therefore as $p\text{A} >=$ or $< q\text{B}$, so is $p\,.\,m\text{A} >=$ or $< q\,.\,m\text{B}$, whatever be the values of p and q, and hence mA : mB :: A : B.

THEOREM 6.

If two magnitudes (A, B) *have the same ratio as two whole numbers* (m, n), *then* $n\text{A} = m\text{B}$: *and conversely if* $n\text{A} = m\text{B}$, A *has to* B *the same ratio as* m *to* n.

Proof. Of A and m take the equimultiples nA and $n\,.\,m$, and of B and n take the equimultiples mB and $m\,.\,n$,
then since A : B :: m : n,
therefore as nA is $>=<$ mB, so is $nm >=< m\,.\,n$,
but since $n\,.\,m = m\,.\,n$, it follows (Def. 4) that $n\text{A} = m\text{B}$.

Again since mB : nB :: m : n we have, if $n\text{A} = m\text{B}$, nA : nB :: m : n; whence it follows (Theor. 5) that A : B :: m : n.

COR. *If* $A : B :: P : Q$ *and* $nA = mB$, *then* $nP = mQ$; *whence if* A *be a multiple, part, or multiple of a part of* B, P *is the same multiple, part, or multiple of a part of* Q.

THEOREM 7.

If four magnitudes of the same kind be proportionals, the first will be greater than, equal to, or less than the third, according as the second is greater than, equal to, or less than the fourth.

Proof. Let $A : B :: C : D$.

Then if $A = C$, $A : B :: C : B$, and therefore $C : D :: C : B$, whence $B = D$.

Also if $A > C$, $A : B > C : B$, and therefore $C : D > C : B$, whence $B > D$.

Again if $A < C$, $A : B < C : B$, and therefore $C : D < C : B$, whence $B < D$.

THEOREM 8.

If four magnitudes of the same kind be proportionals, the first will have to the third the same ratio as the second to the fourth.

Proof. Let $A : B :: C : D$, then $A : C :: B : D$.

For (Th. 6) $mA : mB :: A : B$ and $nC : nD :: C : D$,

therefore $$mA : mB :: nC : nD,$$

whence (Th. 7) $mA >= \text{or} < nC$, as $mB >= \text{or} < nD$,

and this being true for all values of m and n,

$$A : C :: B : D.$$

Theorem 9.

If any number of magnitudes of the same kind be proportionals, as one of the antecedents is to its consequent, so shall the sum of the antecedents be to the sum of the consequents.

Proof. Let $A : B :: C : D :: E : F$, then $A : B :: A + C + E : B + D + F$.

For as $mA > = < nB$, so is $mC > =$ or $< nD$,
and so also is $mE > =$ or $< nF$; whence it follows
that so also is $mA + mC + mE > =$ or $< nB + nD + nF$,
and therefore so is $m(A + C + E) > =$ or $< n(B + D + F)$,
whence $A : B :: A + C + E : B + D + F$.

Theorem 10.

If two ratios are equal, the sum or difference of the antecedent and consequent of the first has to the consequent the same ratio as the sum or difference of the antecedent and consequent of the other has to its consequent.

Proof. Let $A : B :: P : Q$, then $A + B : B :: P + Q : Q$ and $A \sim B : B :: P \sim Q : Q$.

For, m being any whole number, n may be found such that either mA is between nB and $(n+1)B$ or $mA = nB$,
and therefore $mA + mB$ is between $mB + nB$ and
$mB + (n+1)B$ or $= mB + nB$,
but $mA + mB = m(A + B)$ and $mB + nB = (m + n)B$,
therefore $m(A + B)$ is between $(m + n)B$ and $(m + n + 1)B$ or $= (m + n)B$.

But as mA is between nB and $(n + 1)B$ or $= nB$,
so is mP between nQ and $(n + 1)Q$ or $= nQ$;
whence as $m(A + B)$ is between $(m + n)B$ and $(m + n + 1)B$ or $= (m + n)B$,

so is $m(P+Q)$ between $(m+n)Q$ and $(m+n+1)Q$ or $= (m+n)Q$,

and therefore, since m is any whole number whatever,

$$A+B \cdot B : P+Q \cdot Q$$

By like reasoning subtracting mB from mA and nB when $A > B$ and therefore $m < n$, and subtracting mA and nB from mB when $A < B$ and therefore $m > n$, it may be proved that

$$A \sim B \quad B \;.\; P \sim Q : Q.$$

COR. *If two ratios are equal, the sum or difference of the antecedent and consequent of the first has to their difference or sum the same ratio as the sum or difference of the antecedent and consequent of the second has to their difference or sum*

THEOREM 11.

If two ratios are equal, and equimultiples of the antecedents and also of the consequents are taken, the multiple of the first antecedent has to that of its consequent the same ratio as the multiple of the other antecedent has to that of its consequent

Proof Let $A \,.\, B \cdot P \,.\, Q$, then $mA : nB \quad mP \quad nQ$.

For $pm \,.\, A >=$ or $< qn \,.\, B$, as $pm \,.\, P >=$ or $< qn \quad Q$,

and therefore $p \quad mA >=$ or $< q \,.\, nB$, as $p \,.\, mP >=$ or $< q \quad nQ$,

whence, p, q being any numbers whatever,

$$mA \,.\, nB \quad mP \,.\, nQ.$$

THEOREM 12

If there be two sets of magnitudes, such that the first is to the second of the first set as the first to the second of the other set, and the second to the third of the first set as the second to the third of the other, and so on to the last magnitude then the first is to the last of the first set as the first to the last of the other.

Proof. Let the two sets of three magnitudes be A, B, C and P, Q, R,

and let $A : B :: P : Q$ and $B : C :: Q : R$,

then $A : C :: P : R$.

[Lemma.—As $A >=$ or $< C$, so is $P <=$ or $< R$.

For if $A > C$, $A : B > C : B$ and $C : B :: R : Q$,

therefore $P : Q > R : Q$, whence $P > R$.

Similarly if $A = C$ or if $A < C$. Hence the lemma is proved.] By Theor. 6, $mA : mB :: mP : mQ$, and by Theor. 11, $mB : nC :: mQ : nR$, whence by the lemma as $mA >=$ or $< nC$, so is $mP >=$ or $< nR$, and therefore, m and n being any numbers whatever,

$$A : C :: P : R.$$

If there be more magnitudes than three in each set, as A, B, C, D and P, Q, R, S;

then, since $A : B :: P : Q$ and $B : C :: Q : R$,

therefore $A : C :: P : R$; but $C : D :: R : S$,

and therefore $A : D :: P : S$. Q. E. D.

Cor. If $A : B :: Q : R$ and $B : C :: P : Q$, then $A : C :: P : R$.

Proof. Let S be a fourth proportional to Q, R, P,

then $Q : R :: P : S$,

therefore $Q : P :: R : S$, (Th. 8)

and $P : Q :: S : R$. (Th. 3)

Hence $A : B :: P : S$ and $B : C :: S . R$,

therefore $A : C :: P : R$.

Def. 9 If there are any number of magnitudes of the same kind, the first is said to have to the last the ratio *compounded* of the ratios of the first to the second, of the second to the third, and so on to the last magnitude

Def. 10. If there are any number of ratios, and a set of magnitudes is taken such that the ratio of the first to the second is equal to the first ratio, and the ratio of the second to the third is equal to the second ratio, and so on, then the first of the set is said to have to the last the ratio *compounded* of the original ratios

Obs. From these definitions it follows, by Theor 12, that if there be two sets of ratios equal to one another, each to each, the ratio compounded of the ratios of the first set is equal to that compounded of the ratios of the other set.

Also that the ratio compounded of a given ratio and its reciprocal is the ratio of equality.

Def When two ratios are equal, the ratio compounded of them is called the *duplicate* ratio of either of the original ratios

Def When three ratios are equal, the ratio compounded of them is called the *triplicate* ratio of any one of the original ratios

SECTION II.

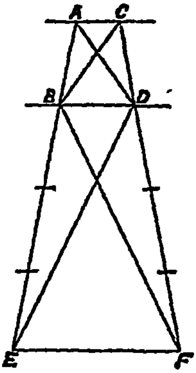
FUNDAMENTAL GEOMETRICAL PROPOSITIONS.

LEMMA.

If on two straight lines AB, CD cut by two parallel straight lines AC, BD equimultiples of the intercepts respectively are taken, then the line joining the points of division will be parallel to AC or BD

Let BE, DF be equimultiples of AB, CD;

Then will EF be parallel to BD.



Proof. Join AD, DE, BC, BF.

Since the triangles ABD, CBD are on the same base BD, and of the same altitude, they are equal.

(II. 2. Cor. 1)

Also whatever multiple BE is of AB, the same multiple is the triangle DBE of the triangle ABD, and the triangle DBF of the triangle CBD:

Therefore the triangle EBD = the triangle FBD, and they are on the same base BD;

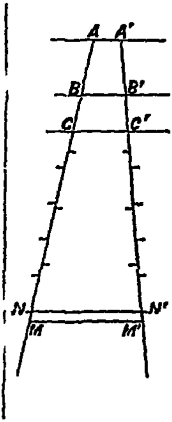
therefore EF is parallel to BD. (II. 2 Cor. 3)

THEOREM I.

If two straight lines are cut by three parallel straight lines, the intercepts on the one are to one another in the same ratio as the corresponding intercepts on the other.

Proof. Let the three parallel lines AA′, BB′, CC′, cut other two lines in A, B, C, and A′, B′, C′ respectively.

then AB : BC :: A′B′ : B′C′.



On the line ABC take BM = m . AB and BN = n . BC, M and N being taken on the same side of B. Also on the line A′B′C′ take B′M′ = m . A′B′ and B′N′ = n . B′C′, M′, N′ being on the same side of B′ as M, N are of B. Then by the Lemma MM′ and NN′ are both parallel to BB′. Hence, whatever be the values of m and n,

as BM (or m . AB) is greater than, equal to, or less than BN (or n . BC),

so is B′M′ (or m . A′B′) greater than, equal to, or less than B′N′ (or n . B′C′),

therefore AB : BC :: A′B′ : B′C′.

It will be observed that the reasoning holds good, whether B be between A and C or beyond A or beyond C.

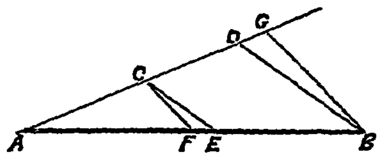
COR. 1. *If the sides of a triangle are cut by a straight line parallel to the base, the segments of one side are to one another in the same ratio as the segments of the other side.*

COR. 2. *If two straight lines are cut by four parallel straight lines the intercepts on the one are to one another in the same ratio as the corresponding intercepts on the other.*

THEOREM 2.

A given finite straight line can be divided internally into segments having any given ratio, and also externally into segments having any given ratio except the ratio of equality: and in each case there is only one such point of division.

Proof. Let AB be the given straight line and, since any given ratio may be expressed as the ratio of two straight

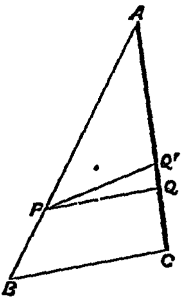


lines, let AC, CD be two lines having the given ratio taken on an indefinite line drawn from A making any angle with AB; join DB; draw CE parallel to DB; then will CE (Theor. 1) divide AB internally in E in the given ratio.
If it could be divided internally at F in the same ratio, BG being drawn parallel to CF to meet AD in G, AF would be to FB as AC to CG, and therefore not as AC to CD. Hence E is the only point which divides AB internally in the given ratio. If CD be taken so that A and D are on the same

side of C, the like construction will determine the external point of division In this case the construction will fail, if CD = AC A like demonstration will shew that there can be only one point of external division in the given ratio.

THEOREM 3

A straight line which divides the sides of a triangle proportionally is parallel to the base of the triangle



Let $AP : PB = AQ : QC$,
then will PQ be parallel to BC

For if not, if possible let some other line PQ' be parallel to BC.

Then $AP : PB = AQ' : Q'C$, (Th 1, Cor 1)

but $AP : PB = AQ : QC$, (Hyp)

Therefore $AQ' : Q'C = AQ : QC$,
which is impossible, (Th. 2)
and therefore PQ is parallel to BC

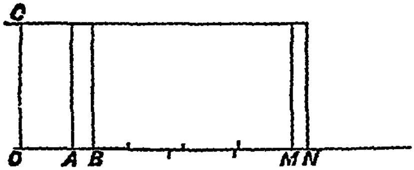
THEOREM 4.

Rectangles of equal altitude are to one another in the same ratio as their bases

Proof Let AC, BC be two rectangles having the common side OC and their bases OA, OB on the same side of OC

In the line OAB indefinitely produced, take OM = m OA and ON = n OB, and complete the rectangles MC and NC

Then MC $= m$. AC and NC $= n$. BC:

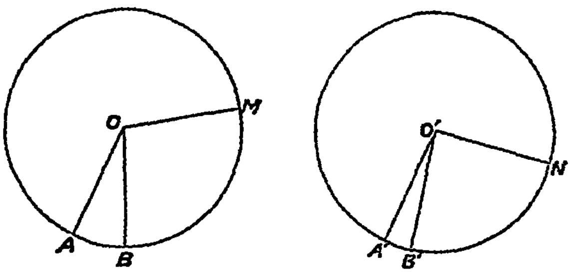


and it is plain that as OM is greater than, equal to, or less than ON, so is MC greater than, equal to, or less than AC; whence the rectangle AC : the rectangle BC :: base OA : base OB.

Cor. *Parallelograms or triangles of the same altitude are to one another as their bases.*

THEOREM 5.

In the same circle or in equal circles angles at the centre and sectors are to one another as the arcs on which they stand.



Proof. Let O, O' be the centres of two equal circles, and let AB, $A'B'$ be any two arcs in them;
then shall the angle or sector AOB be to the angle or sector $A'O'B'$ as the arc AB is to the arc $A'B'$.

For let AM be an arc $= m\ AB$, then the angle or sector between OA and OM (reckoned correspondingly to the arc AM) will be m times the angle or sector AOB.

And let $A'N$ be an arc $= n\ A'B'$, and $A'O'N$ an angle or sector n times the angle or sector $A'O'B'$.

and according as AM is $>$, $=$, or $<$ $A'N$,

so is the angle or sector AOM $>$, $=$, or $<$ the angle or sector $A'O'N$;

and therefore as AB . $A'B'$: angle or sector AOB : angle or sector $A'O'B'$.

BOOK V.

PROPORTION.

INTRODUCTION.

[For the use of those for whom it may be thought well to defer the study of the complete, but more difficult, mode of treatment of Proportion in Book IV, the following Definitions and Propositions referred to in that Book are here collected, with an indication of the principles of an incomplete mode of treatment by which they may be established for commensurable magnitudes.]

Def. 1. One magnitude is said to be a *multiple* of another magnitude when the former contains the latter an exact number of times. According as the number of times is 1, 2, 3.. m, so is the multiple said to be the 1st, 2nd, 3rd ..mth.

Def. 2. One magnitude is said to be a *measure* or *part* of another magnitude when the former is contained an exact number of times in the latter.

Def. 3. If a magnitude can be found which is a measure of two or more magnitudes, these magnitudes are said to be *commensurable*, and the first magnitude is said to be a *common measure* of the others.

It is easy to prove that commensurable magnitudes have also a common multiple, and conversely that magnitudes which have a common multiple are commensurable.

A *measure* of a line is any line which is contained in it an exact number of times Thus an inch is a measure of a foot, and a yard is a measure of a mile So too the measure of an area is any area which is contained an exact number of times in it. A square inch is thus a measure of a square yard *A measure is therefore an aliquot part of any magnitude which it measures* The length of a line, the extent of an area, or any other magnitude, is completely known when we know a measure of it, and how many times it contains that measure.

In measuring any magnitude we take some standard to measure by Thus in measuring length we take a yard, or a foot, or an inch. In measuring solids we take a cubic inch, a cubic foot, or the like. The standard so taken is called the *unit* It may be a precise measure of the magnitude measured, or it may not. The number, whether whole or fractional, which expresses how many times a magnitude contains a certain unit is called the *numerical value* of that magnitude in terms of that unit. Thus in speaking of a line as 7 yards long, a yard is the unit of length, and the numerical value of the line in terms of that unit is 7

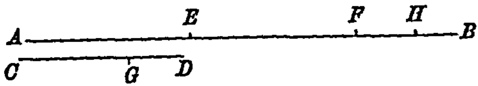
Two lines or magnitudes of the same kind are said to have a *common measure* when there exists a unit of which they can both be expressed as multiples. Thus 15 inches and 1 foot have a common measure, for with the unit 3 inches, their numerical values would be 5 and 4; and with the unit 1 inch their numerical values would be 15 and 12 All whole numbers have unity as a common measure

The following problem gives a method of finding the greatest common measure of two magnitudes, if any common measure exists, and illustrates the familiar Arithmetical method

Problem

To find the greatest common measure of two magnitudes, if they have a common measure

Let AB and CD be the two magnitudes From AB the greater cut



off parts, AE, EF each equal to CD the less, leaving a remainder FB which is less than CD.

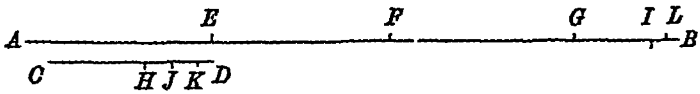
From CD cut off parts, CG..., equal to FB, leaving a remainder GD less than FB.

From FB cut off parts FH, HB.. equal to GD: and continue this process until a remainder GD is found which is contained *an exact number* of times in the previous remainder, so that no further remainder is left. The last remainder is then the greatest common measure.

For, firstly, since GD measures FB, it also measures CG; and therefore measures CD But $CD=AE$ and EF; and therefore GD measures AE, EF and FB, that is it measures AB. Hence GD is *a* common measure of AB and CD.

And again, since every measure of CD and AB must measure AF, it must measure FB or CG, and therefore also GD· hence the common measure cannot be greater than GD; that is GD is the *greatest* common measure.

So also, in the figure adjoining, the first remainder is GB, the



second HD; the third IB, the fourth KD, which is contained exactly twice in IB Hence KD is the greatest common measure, and it will be seen to be contained twice in IB, and therefore five times in HD, seven times in GB, 12 times in CD, and 43 times in AB.

Hence AB and CD have as their numerical values 43 and 12 in terms of the unit KD.

Cor. Every measure of KD is a common measure of AB and CD.

When magnitudes have a common measure they are called *commensurable.* But it is very frequently the case in Geometrical figures, that lines and other magnitudes have no common measure; the process above given continuing indefinitely; the remainder becoming smaller at each step of the process but never actually disappearing. In this case the lines are said to be *incommensurable.*

Def 4. The *ratio* of one magnitude to another of the same kind is the relation of the former to the latter in respect of *quantuplicity*

The ratio of A to B is denoted thus, A   B, and A is called the *antecedent*, B the *consequent*

The complete examination of the nature of the comparison of two magnitudes according to quantuplicity is contained in Book IV For numbers, and for magnitudes generally, *so far as they are commensurable* (and it is to be noted that this is not the *normal*, but the *exceptional*, case), the comparison may be made in a more simple manner either

(1) (As is usual in Arithmetic) by considering what multiple, part, or multiple of a part one magnitude is of the other,
or (2) by considering what multiples of the two magnitudes are equal to one another]

Def 5 When the ratio A . B is equal to the ratio P   Q, *i e* either

(1) When A is the same multiple, part, or multiple of a part of B as P is of Q, or,

(2) When like multiples of A and P are equal respectively to like multiples of B and Q,
the four magnitudes are said to be *proportionals*, or to form *proportion*

The equality of the ratios is denoted by the symbol   , and the proportion thus, A   B . P   Q, which is read A is to B as P is to Q

A and Q are called the *extremes*, B and P the *means*, and Q is said to be the *fourth proportional* to A, B and P   The antecedents A, P are said to be *homologous* to one another, and so also are the consequents

Def. 6. If A, B, C are three magnitudes of the same kind such that A : B :: B : C, B is said to be the *mean proportional* between A and C, and C the *third proportional* to A and B.

Def. 7. If there are two ratios A : B, P : Q, and C be taken such that B : C P : Q, then A is said to have to C a ratio *compounded* of the ratios A : B, P : Q. Thus if there are three magnitudes A, B, C, then A has to C the ratio compounded of the ratios A : B, B : C.

Def. 8 A ratio compounded of two equal ratios is called the *duplicate* of either of these ratios.

It is evident that different ratios cannot have the same duplicate ratio

General Propositions on Proportion.

[All these propositions admit of obvious algebraical proof]

(1.) Ratios that are equal to the same ratio are equal to one another.

(2) Equal magnitudes have the same ratio to the same or to equal magnitudes.

(3) Magnitudes that have the same ratio to the same or equal magnitudes are equal.

(4.) The ratio of two magnitudes is equal to that of their halves or doubles.

(5.) If A : B .. P : Q, then B : A :· Q : P.

(invertendo)

(6.) If A : B :: C : D, all the four being of the same kind,

then A : C :: B : D. (alternando)

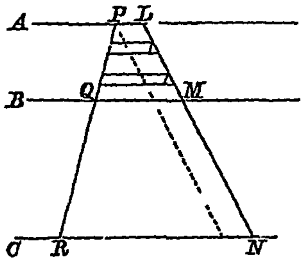
(7) If $A : B :: P : Q$,
then $A+B : B :: P+Q : Q$, (componendo)
and $A-B : B :: P-Q : Q$. (dividendo)

(8) If $A : B :: C : D :: E : F$,
then $A+C+E : B+D+F :: A : B$. (addendo)

(9) If $A : B :: P : Q$
and $B : C :: Q : R$,
then $A : C :: P : R$. (ex æquali)

Theorem 1

If two straight lines are cut by three parallel straight lines, the intercepts on the one are to one another in the same ratio as the corresponding intercepts on the other.



Let A, B, C be the three parallels, PQR, LMN any two lines intersected by them; then shall

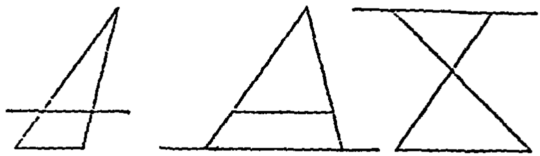
$$PQ : QR :: LM : MN.$$

Proof. Let PQ, QR be commensurable, and contain their common measure m and n times respectively: and through the points of division let lines parallel to A be drawn to meet LM.

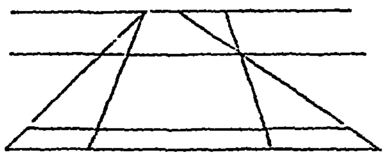
Then (by I. 32) LM and MN will be divided into m and n equal parts respectively,

therefore $$PQ : QR :: m : n$$
$$:: LM \quad MN$$

Cor. 1. *If the sides of a triangle are cut by a straight line parallel to the base, the segments of one side are to one another in the same ratio as the segments of the other side*



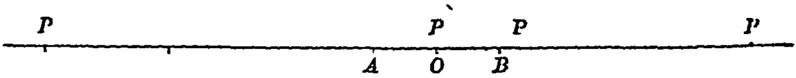
Cor 2. *If two straight lines are cut by four parallel straight lines the intercepts on the one are to one another in the same ratio as the corresponding intercepts on the other.*



Theorem 2.

A given finite straight line can be divided internally into segments having any given ratio, and also externally into segments having any given ratio except the ratio of equality: and in each case there is only one such point of division.

Let AB be the given finite straight line, and let O be the point of bisection of AB, then if P is at O the ratio $\frac{PA}{PB} = 1$.



Conceive the point P to move to the right towards B, then the ratio $\frac{PA}{PB}$ continually increases until, when P approaches indefinitely near to B the ratio becomes infinite; and for intermediate positions it has passed continuously through every value between 1 and ∞ (infinity)

When P is at the right of B the ratio

$$\frac{PA}{PB} = \frac{PB + AB}{PB} = 1 + \frac{AB}{PB},$$

and is therefore greater than 1

When PB is very small $\frac{AB}{PB}$ is very large, and as PB increases $\frac{AB}{PB}$ diminishes until it becomes indefinitely small, and therefore $\frac{PA}{PB}$ becomes as nearly equal to 1 as we please, and has passed continuously through every value between ∞ and 1.

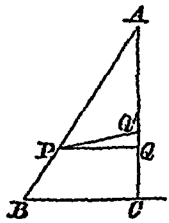
Hence for any assigned value of the ratio greater than 1 there are two positions for P, one between O and B, and one to the right of B

Similarly as P moves from O to A, $\frac{PA}{PB}$ passes through every value from 1 to 0, and as it moves to the left of A it

passes through every value from 0 to 1, and therefore for every value of the ratio less than 1 there are two positions for P, one between O and A, and one to the left of A.

THEOREM 3.

A straight line which divides the sides of a triangle proportionally is parallel to the base of the triangle.



For let $AP : PB :: AQ : QC$, and suppose PQ not parallel to BC, but if possible let PQ' be parallel to BC;

then $$AP : PB :: AQ' : Q'C,$$

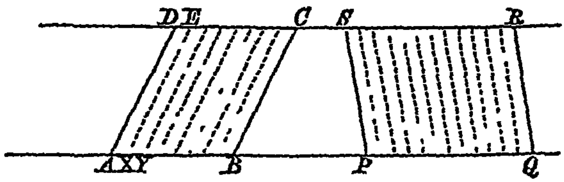
and therefore $$AQ : QC :: AQ' : Q'C,$$

which is impossible by Theorem 2.

THEOREM 4.

Parallelograms of the same altitude are to one another as their bases.

Let $ABCD$, $PQRS$ be parallelograms of the same altitude on the bases AB, PQ.



Then shall $DABC$ be to $SPQR$ as AB to PQ.

Let AB, PQ be commensurable, and let them contain their common measure m and n times respectively. Through the points of division draw lines parallel to the sides of the parallelogram. Then the parallelograms will be divided into m and n equal parts respectively, (II 1 Cor 2)

and therefore $$DABC : SPQR \quad m \cdot n$$
$$: AB \quad PQ$$

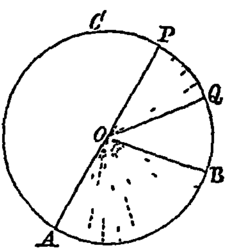
COR. 1. *Triangles of the same altitude are to one another as their bases*

For a triangle is half the parallelogram on the same base and having the same altitude as the triangle.

THEOREM 5.

In the same circle or in equal circles angles at the centre and sectors are to one another as the arcs on which they stand

Let ABC be a circle, of which O is the centre



And let AOB, POQ be two angles at the centre

Then $\angle AOB \,.\, POQ \quad \text{arc } AB \quad \text{arc } PQ$

$\text{sector } AOB : \text{sector } POQ.$

Let the angles AOB, POQ be commensurable, and let them contain their common measure m and n times respectively; and let the angles be divided into equal parts by radii.

Then (iii. 2) the areas and sectors are also divided into m and n equal parts respectively,

and therefore

$$\text{arc } AB : \text{arc } PQ :: m : n$$
$$:: \angle AOB : \angle POQ$$
$$:: \text{sector } AOB : \text{sector } POQ.$$

SECTION I

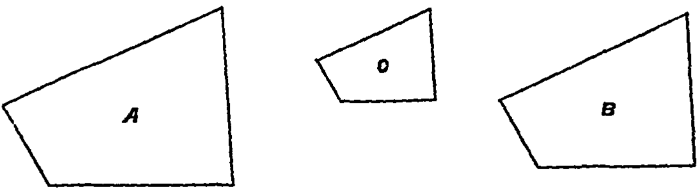
SIMILAR FIGURES

Def 1. Similar rectilineal figures are those which have their angles equal, and the sides about the equal angles proportional

Def 2 Similar figures are said to be *similarly described upon given straight lines*, when those straight lines are homologous sides of the figures

THEOREM 1

Rectilineal figures that are similar to the same rectilineal figure are similar to one another.



Proof. Let A, B be each of them similar to C, then will A be similar to B

Proof. Since the angles of A and B are respectively equal to the angles of C,

therefore also A and B are equiangular·

and since the sides about each angle of A are in the same ratio as the sides about the equal angle of C; and the sides about each of B are also in the same ratio,

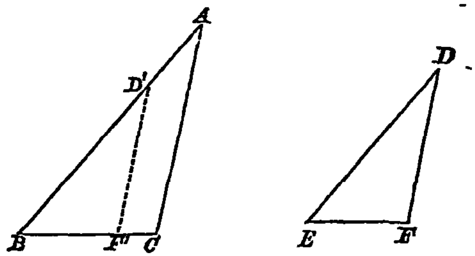
therefore the sides about the equal angles of A and B are proportionals;

therefore A is similar to B (v. Def I.)

THEOREM 2

If two triangles have their angles respectively equal, they are similar, and those sides which are opposite to the equal angles are homologous

Let ABC. DEF be two triangles, which have the angles of the one equal to the angles of the other, viz. A, B, C respectively equal to D, E, F respectively;



Then shall they be similar, that is

$$AB : BC :: DE : EF,$$

and $$BC : CA :: EF : FD,$$

and $$CA : AB :: FD : DE.$$

Conceive the angle E placed on the angle B, then F and D would fall as F' and D' on BC and BA, or on those lines produced and because the $\angle F' =$ the $\angle C$, therefore $F'D'$ is parallel to CA,

and therefore $\quad BF' \quad BC \quad BD' \quad BA,$ (IV. 1)

and therefore $\quad BF' : BD' \quad BC \;.\; BA,$

that is $\quad EF : ED \quad BC \quad BA$

Similarly by placing F on C, and D on A, the other proportions are obtained, and therefore the triangles are similar

This theorem is a generalization of Theorem 15 in Book 1 *If two angles and a side of one triangle are respectively equal to two angles and the corresponding side of another triangle, these triangles will be equal in all respects.*

THEOREM 3

If two triangles have one angle of the one equal to one angle of the other and the sides about these angles proportional, they are similar, and those angles which are opposite to the homologous sides are equal

Let the triangles ABC, DEF have the angles at B and E equal, and let $BA \;.\; BC \;.\; ED \quad EF$, then will the triangles be similar

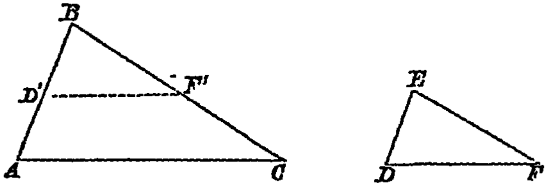
Conceive the angle E placed on the equal angle B, then D and F will fall as at D' and F' on the sides BA, BC,

and since $\quad BA \quad BC \quad ED \quad EF,$

therefore $\quad BA \quad BD' \quad BC \;.\; BF',$

and therefore $D'F'$ is parallel to AC, (IV 3)

and the angles $BD'F'$ and BFD', that is, D and F, are



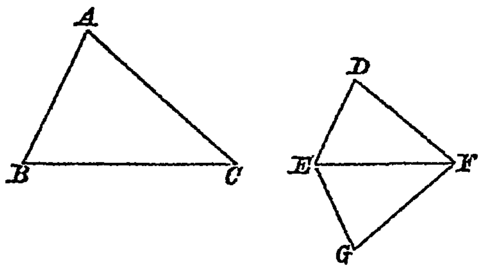
equal respectively to the angles A and C. Hence the triangles are equiangular and therefore similar.

This theorem is a generalization of Book I. Theorem 16 *If two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another, the triangles will be equal in all respects.*

THEOREM 4.

If two triangles have the sides taken in order about each of their angles proportional, they are similar, and those angles which are opposite to the homologous sides are equal.

Let ABC, DEF be two triangles which have their sides



about each of their angles proportional,

that is, $AB : BC :: DE : EF$,

and $BC : CA :: EF : FD$,

and therefore also $CA : AB :: FD : DE$;

then will the triangles ABC, DEF be similar.

Conceive a triangle equiangular to ABC applied to EF, on the opposite side of the base EF, so that the angles FEG, EFG are equal to B and C respectively

Then since the triangle GEF is equiangular to ABC, it is therefore similar,

and therefore $\quad GE \,.\, EF \cdot AB : BC,$
but $\quad AB : BC \quad DE \,.\, EF,$
and therefore $\quad GE \,.\, EF \cdot DE \cdot EF,$
and therefore $\quad GE = ED.$
Similarly $\quad GF = DF,$
and the triangle DEF is therefore equiangular to GEF, (I. 18) and therefore also to ABC

Therefore the triangle DEF is similar to the triangle ABC.

This theorem is a generalization of Book I Theorem 18 *If the three sides of one triangle are respectively equal to the three sides of another, these triangles will be equal in all respects.*

THEOREM 5

If two triangles have one angle of the one equal to one angle of the other, and the sides about one other angle in each proportional, so that the sides opposite the equal angles are homologous, the triangles have their third angles either equal or supplementary and in the former case the triangles are similar.

Let ABC, DEF be the two triangles having the angle $B =$ the angle E,

and $\quad BA \,.\, AC \cdot ED \quad DF,$

then will the angle C be equal or supplementary to the angle F.

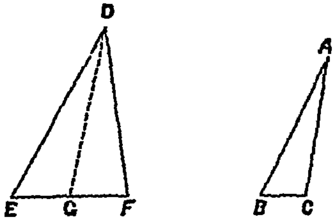
Proof. The angle A is equal or unequal to the angle D.

If $A = D$ (fig. 1), then, by Th. 3, the triangles are similar, and the angle C is *equal* to the angle F.

If A is not equal to D, as in fig 2, at the point D make the angle $EDG = BAC$;

Fig (1)

Fig (2)



then the triangle DEG is equiangular and similar to the triangle ABC,

and therefore $\quad ED : DG :: BA : AC.$

But $\quad ED : DF :: BA : AC,$

and therefore $\quad DG = DF,$

and therefore the angle $DGF =$ the angle DFG,

and because the angle $EGD =$ the angle C,

and EGD is supplementary to DGF;

therefore C is supplementary to DFE.

Hence the angle C is either equal or supplementary to the angle F, and in the former case the triangles are similar.

This theorem is a generalization of Book 1. Theorem 20.

COR. 1. *If the two angles given equal are right angles or obtuse angles, the remaining angles must be both acute, and therefore cannot be supplementary, and are therefore equal, and the triangles are similar.*

COR. 2. *If the angles opposite to the other two homologous sides are both acute or both obtuse, or if one of them is a right angle, then these angles must be equal, and the triangles are similar.*

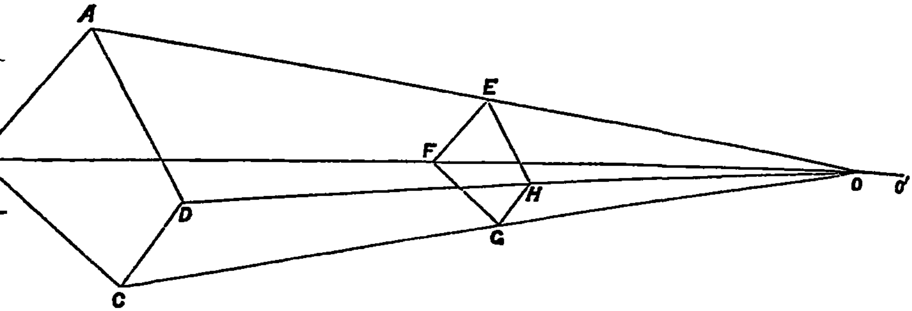
COR. 3. *If the side opposite the given angle in each triangle is not less than the other given side, then the given angles must be not less than the third angles: therefore the third angles must be both acute, and therefore cannot be supplementary. They are therefore equal and the triangles are similar.*

THEOREM 6.

If two similar rectilineal figures are placed so as to have their corresponding sides parallel, all the straight lines joining the angular points of the one to the corresponding angular points of the other are parallel or meet in a point; and the distances from that point along any straight line to the points where it meets corresponding sides of the figures are in the ratio of the corresponding sides of the figures.

Let $ABCD$, $EFGH$ be the two rectilinear figures

Let AB, BC be two consecutive sides of the rectilineal figure $ABCD$, and EF, FG the corresponding sides of the



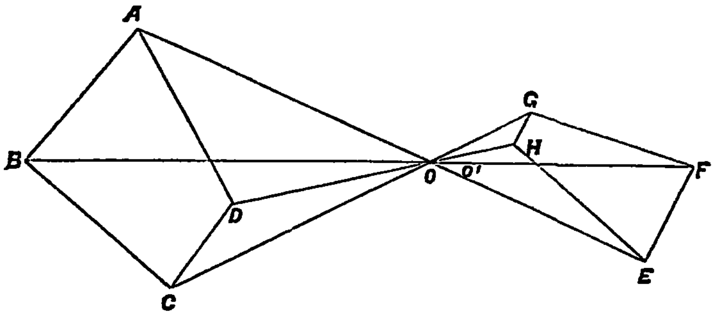


figure $EFGH$. Let AE, BF meet in O. it is required to prove that CG passes through O.

If not let it cut BF in some other point O'.

Then by the similar triangles ABO, EFO,

$$AB : EF : BO : FO;$$

also by the similar triangles BCO', FGO',

$$BC : FG :: BO' : FO'.$$

But $$AB : EF : BC : FG$$

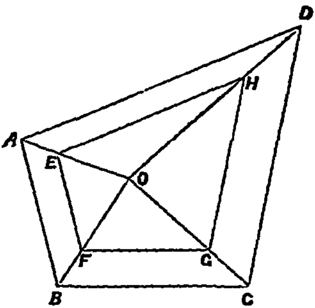
by hypothesis, since the rectilinear figures are similar;

therefore $$BO . FO \quad BO' : FO',$$

and therefore the points O, O' coincide. Bk IV 2

$\therefore$ CG does pass through O and in the same manner DH passes through O

COR. *Similar rectilineal figures may be divided into the same number of similar triangles*



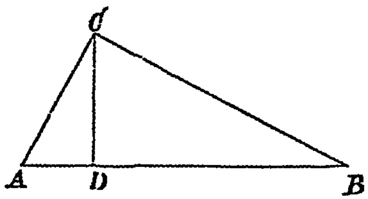
For if one rectilineal figure were placed within the other, and with their corresponding sides parallel, as in the figure, the lines AE, BF, CG, DH would all meet in one point O, and the triangles into which the polygons are respectively divided are similar

Def 3. The point determined as in Theor. 6 is called a *centre of similarity* of the two rectilineal figures

THEOREM 7

In a right-angled triangle if a perpendicular is drawn from the right angle to the hypotenuse it divides the triangle into two other triangles which are similar to the whole and to one another

Let ACB be the triangle, right-angled at C, CD the perpendicular.



Then the triangles CAD, BAC have two angles CAD and CDA of the one equal respectively to BAC, BCA of the other; therefore they are equiangular, and similar.

In the same manner DCB is equiangular and similar to either DAC or CAB.

Cor. $AD : DC :: DC : DB$,

or the perpendicular from the right angle of a right-angled triangle on the hypothenuse is a mean proportional between the segments of the base

Also $BA : AC :: AC : AD$,

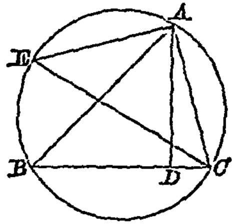
and $AB : BC :: BC : BD$,

or the side of a right-angled triangle is a mean proportional between the hypothenuse and the projection on it of that side.

Theorem 8.

If from any angle of a triangle a straight line is drawn perpendicular to the base, the diameter of the circle circumscribing the triangle is a fourth proportional to the perpendicular and the sides of the triangle which contain that angle.

Let ABC be a triangle, and let AD be drawn from the angle A perpendicular to the base BC, and let CE be the diameter of the circle circumscribing the triangle ABC;



then shall

$$AD \cdot AB \quad AC \quad CE$$

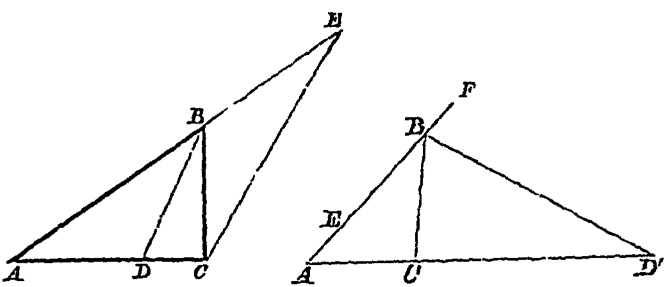
Proof Join AE.

Then the triangles ADB, CAE have the angles $ABD = CEA$ in the same segment, and $ADB = CAE$ being right angles, therefore they are similar, and therefore

$$AD \quad AB \quad AC \quad CE$$

✓ THEOREM 9

If the interior or exterior vertical angle of a triangle is bisected by a straight line which also cuts the base, the base is divided internally or externally in the ratio of the sides of the triangle And, conversely, if the base is divided internally or externally in the ratio of the sides of the triangle, the straight line drawn from the point of division to the vertex bisects the interior or exterior vertical angle



Let ABC be a triangle, BD the bisector of the angle ABC.

Then will $\quad AD : DC :: AB : BC.$

Draw CE parallel to BD to meet AB produced.

Then by parallelism the angle BCE = the angle DBC, and the angle BEC = the angle ABD or FBD.' But ABD or $FBD = DBC$, and therefore the angle BCE = the angle BEC; and therefore $BE = BC$.

But because AE, AC are cut by the parallels DB, CE;

therefore $\quad AD : DC :: AB : BE,$ IV 2

that is, $\quad AD : DC :: AB : BC$

COR. 1. *Conversely, if $AD : DC :: AB : BC$, then BD is the bisector of the angle ABC, or of CBF, according as AC is divided internally or externally in D.*

For there is only one internal bisector of the angle, and only one point D which divides the base internally or externally, so that

$$AD : DC :: AB : BC;$$

and therefore, since the bisector divides the base in this ratio, the line which divides the base in this ratio is the bisector.

(This may also be proved directly)

COR. 2. *If* AB = BC, *then the ratio of* AD′ : D′C *becomes* = 1, *which indicates that* D′ *is at an infinite distance (by* IV. 2) *Hence the external bisector of the vertical angle of an isosceles triangle is parallel to the base*

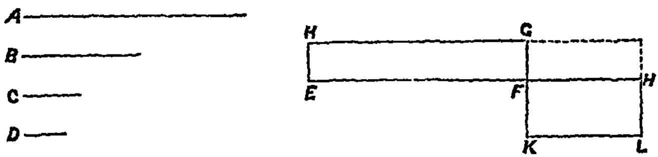
COR. 3. *If* B *moves so that the ratio* AB : BC *is constant, the bisectors of the interior and exterior angles will always pass through the fixed points* D, D′ *which divide* AC *internally and externally in that ratio.*

SECTION II

AREAS

THEOREM 10

If four straight lines are proportional the rectangle contained by the extremes is equal to the rectangle contained by the means, and, conversely, if the rectangle contained by the extremes is equal to the rectangle contained by the means the four straight lines are proportional



Let A be to B as C to D,
then will the rectangle contained by A and D be equal to the rectangle contained by C and D.

Proof Construct the rectangle $EFGH$, with the sides $EF = A$, $FG = D$, and also the rectangle $HFKL$, with the sides $HF = B$, $FK = C$, and place them so that EF, FH are in one straight line, and therefore also GF, FK in one straight line

Complete the rectangle GH

Then rectangle EG rect. GH EF FH, (IV 4)

A B,

and $\quad$ rect. KH : rect. GH :: KF : GF,

$\quad$ :: C : D,

but $\quad$ EQ $\quad$ $A : B :: C : D$,

therefore rect. FG : rect. GH :: rect KH : rect. GH,

therefore $\quad$ rect. EG = rect. GH, KH

that is, the rectangle contained by A and D is equal to the rectangle contained by C and D.

Conversely, if the rectangle contained by A and D is equal to the rectangle contained by B and C,

then $\quad$ A $\quad$ B $\quad$ $C \cdot D$

Proof. The same construction being made,

because $\quad$ rect. EG = rect. KH, $\quad$ (Hyp)

therefore rect. EG $\quad$ rect. GH . rect KH : rect. GH,

but $\quad$ rect. EG : rect. GH $\quad$ EF . FH,

$\quad$. A . B,

and $\quad$ rect. KH $\quad$ rect GH . KF : FG,

$\quad$. C : D,

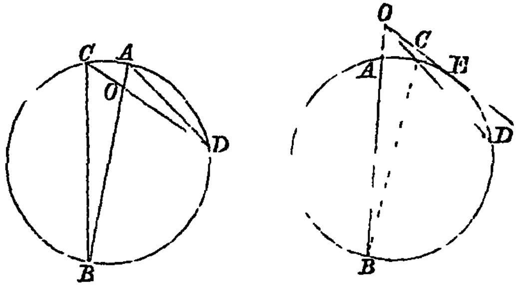
therefore $\quad$ $A \cdot B$ $\quad$ $C : D$.

COR. *If three straight lines are proportional the rectangle contained by the extremes is equal to the square on the mean; and, conversely, if the rectangle contained by the extremes of three straight lines is equal to the square on the mean the lines are proportional*

THEOREM 11.

If two chords of a circle intersect either within or without a circle the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

Let AOB, COD be the chords through O. Then is $AO \times OB = CO \times OD$.



For join CB, AD Then since the angle D = angle B in the same segment, and the angle at O common to the two triangles AOD, BOC, the triangles are equiangular and similar,

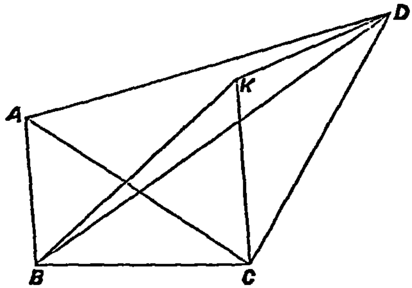
$$\therefore AO : OD :: CO : OB,$$
$$\therefore AO \times OB = CO \times OD.$$

COR. If one of the secants OCD, in figure 2, become a tangent, as OE, then OC and OD become equal to OE, and therefore $AO : OE :: OE : OB$, and $OE^2 = AO \times OB$, or *the square on the tangent to a circle from any point is equal to the rectangle contained by the intercepts on the secant drawn from that point*

THEOREM 12

The rectangle contained by the diagonals of a quadrilateral is less than the sum of the rectangles contained by opposite sides unless a circle can be circumscribed about a quadrilateral, in which case it is equal to that sum

Let $ABCD$ be a quadrilateral figure · then will the rectangle contained by the diagonals AC, BD be less than the sum of the rectangles contained by AB, CD and by AD, CB respectively, unless a circle can be described about $ABCD$.



Proof. At the point C in the straight line CD make the angle DCK equal to the angle ACB, and therefore also BCK equal to ACD; and at the point D make the angle CDK equal to the angle BAC.

Join BK.

Then the triangle CDK is similar to the triangle CAB by construction;

and therefore $\quad AB : AC :: DK : DC$;

therefore the rectangle $AB \times DC =$ rect. $AC \times DK$.

Again, because the triangles CDK, CAB are similar,

$$BC : CA \quad KC : CD,$$

and therefore $\quad BC \cdot KC \cdot CA : CD$;

and the angle BCK is equal to the angle ACD, (Constr)

therefore the triangles BCK, ACD are similar, (Th 3)

and $\quad BC : BK \quad AC : AD$;

therefore the rect. $BC \times AD =$ rect. $AC \times BK$;

but it was proved that the rect $AB \times DC$ = rect $CA \times DK$, therefore the sum of the rectangles $BC \times AD + AB \times DC$ = rectangle contained by AC and the sum of KB and KD

But the sum of KB and KD is greater than BD,

therefore $BC \times AD + AB \times DC$ is greater than $CA \times BD$

But if the quadrilateral $ABCD$ could have a circle described about it,

then the angle CDB would be equal to the angle CAB in the same segment;

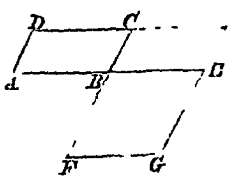
and therefore the point K would fall on BD,

and therefore $BK + KD = BD$

In this case therefore the rectangle contained by the diagonals is equal to the sum of the rectangles contained by the opposite sides of the quadrilateral

THEOREM 13

If two triangles or parallelograms have one angle of the one equal to one angle of the other, their areas have to one another the ratio compounded of the ratios of the including sides of the first to the including sides of the second.



Let $ABCD$, $EBFG$ be the parallelograms, and let them be placed so as to have AB, BE in one straight line, and therefore also, since the parallelograms are equiangular, so as to have CB, BF in one straight line

Complete the parallelogram CBE.

Then the ratio of $DB \cdot BG$ is compounded of the ratios of $DB : CE$ and of CE to BG.

But $\qquad DB : CE :: AB : BE$,

and $\qquad CE : BG :: CB : BF$;

therefore, the ratio of $DB : BG$ is compounded of the ratios $AB \,.\, BE$ and $CB : BF$.

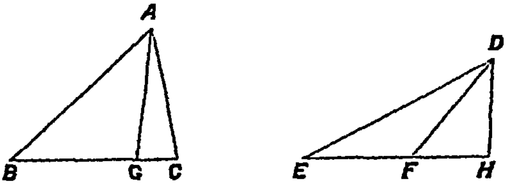
Cor. 1. *If two triangles or parallelograms have one angle of the one supplementary to one angle of the other, their areas have to one another the ratio compounded of the ratios of the including sides of the first to the including sides of the second.*

Cor. 2. *The ratio compounded of two ratios between straight lines is the same as the ratio of the rectangle contained by the antecedents to the rectangle contained by the consequents.*

Theorem 14.

Triangles and parallelograms have to one another the ratio compounded of the ratios of their bases and of their altitudes.

Let ABC, DEF be two triangles, having the altitudes AG, DH respectively;



then shall the triangle ABC have to the triangle DEF the

ratio compounded of the ratios of the bases BC to EF, and of the altitudes AG to DH

Proof The triangle ABC is half the rectangle contained by AG and BC, (II Th 2)
and the triangle DEF is half the rectangle contained by DH and EF,

therefore the triangle ABC triangle DEF
rectangle AG, BC rect DH, EF,

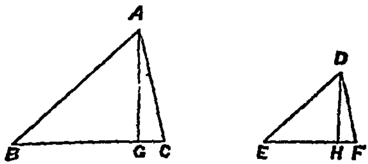
that is in the ratio compounded of AG DH and of $BC : EF$ (Th 13 Cor 2)

In the same manner it may be shewn that parallelograms are to one another in the ratio compounded of the ratios of their bases and of their altitudes

THEOREM 15

Similar triangles are to one another in the duplicate ratio of their homologous sides

Let ABC, DEF be similar triangles having the angles at A, B, C respectively equal to the angles at D, E, F,



then shall the triangles be to one another in the duplicate ratio of $BC : EF$

Proof Let fall the perpendiculars AG, DH to the sides BC, EF,
then, by the last theorem,

the triangle ABC is to the triangle DEF in the ratio compounded of the ratios of AG to DH and of BC to EF,

but $AG \cdot DH \,.\, AB \cdot DE$, by similar triangles,

and $AB : DE \cdot BC : EF$; (Hyp)

therefore $AG \,.\, DH \,.\, BC : EF$,

and therefore the ratio compounded of the ratios of AG DH and of $BC \,.\, EF$ is equal to the ratio compounded of the ratios of $BC \,.\, EF$ and of $BC \,.\, EF$,

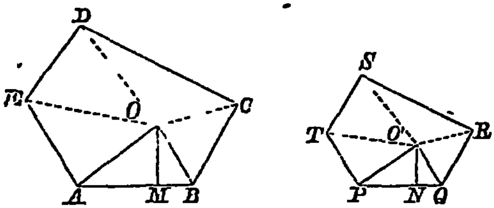
that is to the duplicate ratio of $BC \cdot EF$, (4 Def. 8)

therefore the triangle ABC is to the triangle DEF in the duplicate ratio of BC to EF.

THEOREM 16

The areas of similar rectilineal figures are to one another in the duplicate ratio of their homologous sides

Let $ABCDE$, $PQRST$ be similar polygons



Divide each of them into the same number of similar triangles by lines drawn from the points O, O'. (v 6 Cor)

Let OAB, $O'PQ$ be two similar triangles

Then the triangle OAB is to the triangle $O'PQ$ in the duplicate ratio of $AB : PQ$, (Th 15)

and the triangle AOE is to the triangle $PO'R$ in the duplicate ratio of AE to PR.

but $AB \,.\, PQ \cdot AE \,.\, PR$,

because the polygons are similar.

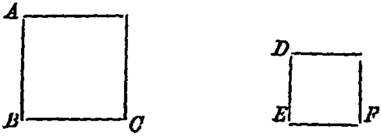
and therefore the triangle AOE is to the triangle $O'PQ$ in the duplicate ratio of AB to PQ

Similarly it may be proved that each of the triangles into which $ABCDE$ is divided is to the corresponding triangle of those into which $PQRST$ is divided in the duplicate ratio of $AB : PQ$

Therefore the polygon $ABCDE$ is to the polygon $PQRST$ in the duplicate ratio of AB PQ

COR. 1. *Similar rectilineal figures are to one another as the squares described on their homologous sides.*

For if ABC, DEF are squares, then by the theorem AC DF is the duplicate ratio of BC EF



But any similar polygons similarly described on BC and EF are to one another in the duplicate ratio of BC to EF, therefore they are in the ratio of the squares on BC and EF

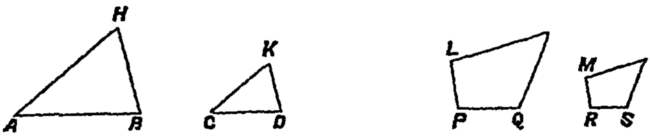
Hence, further, if three straight lines are in continued proportion, the first is to the third in the duplicate ratio of the first to the second,

but the squares on the first and second are in the duplicate ratio of the first to the second,

therefore the first is to the third, as the square described on the first is to the square described on the second:

and therefore, further, if three straight lines be in continued proportion, the 1st : 3rd as any polygon described on the 1st : the similar and similarly described polygon on the 2nd

Cor 2 *If four straight lines are proportional and a pair of similar rectilineal figures are similarly described on the first and second, and also a pair on the third and fourth, these figures are proportional, and conversely, if a rectilineal figure on the first of four straight lines is to the similar and similarly described figure on the second as a rectilineal figure on the third is to the similar and similarly described figure on the fourth, the four straight lines are proportional*



Let $AB : CD :: PQ : RS$,

and let the figures ABH, CDK, and likewise LPQ, MRS, be respectively similar, and similarly situated on AB, CD, PQ, RS,

then $ABH : CDK :: LPQ : MRS$

Proof Since $AB : CD :: PQ : RS$,

therefore the duplicate ratio of $AB : CD$ is equal to the duplicate ratio of $PQ : RS$

But $HAB : KCD$ in the duplicate ratio of $AB : CD$, and $LPQ : MRS$ in the duplicate ratio of $PQ : RS$,

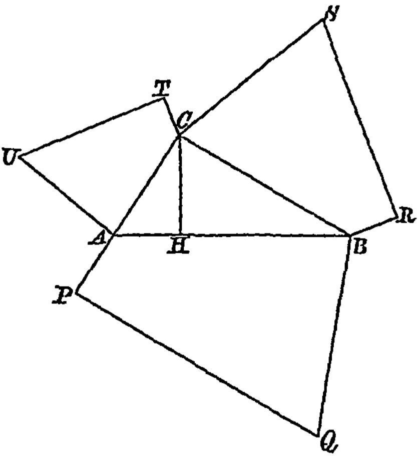
therefore $HAB : KCD :: LPQ : MRS$

The converse follows by the rule of identity.

THEOREM 17

In any right-angled triangle, any rectilineal figure described on the hypothenuse is equal to the sum of two similar and similarly described figures on the sides.

Let ABC be a triangle right-angled at C, and let $APQB$, $BRSC$, $CTUA$ be similar figures similarly de-



scribed on the sides AB, BC, CA, that is, figures of which AB, BC, CA are homologous sides

Then will $\quad APQB = BRSC + CTUA$

Draw CH perpendicular to AB

Then $AB \,.\, BC \quad BC \quad BH$ by similar triangles, and therefore (v 7 Cor)

$APQB \quad BRSC \quad AB \,.\, BH$ (v 14 Cor 3) in the same manner it may be shewn that

$$APQB \quad CTUA \quad AB \quad AH,$$

and therefore

$$APQB : BRSC + CTUA \quad AB : BH + AH;$$

but

$$AB = BH + AH;$$

and therefore $\quad APQB = QRSC + CTUA.$

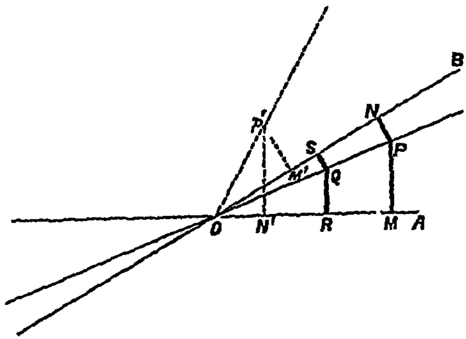
It is obvious that a special case of this theorem is the theorem proved before, that the square on the hypothenuse of a right-angled triangle is equal to the sum of the squares on the sides.

SECTION III

LOCI AND PROBLEMS.

LOCI.

i *The locus of a point whose distances from two fixed straight lines are in a constant ratio is a pair of straight lines, passing through the point of intersection of the given lines, if they intersect, and parallel to them, if the lines are parallel*



First, let OA, OB intersect in O, and let P be one of the points on the locus, so that PM PN in the given ratio

Join OP, and let Q be any point in OP

Draw QR, QS perpendicular to OA, OB

Then Q will be a point on the locus

For by similar triangles OQR, OPM,

$$PM : QR \quad OP . OQ,$$

and for the same reason

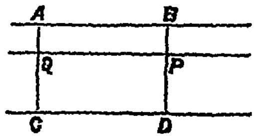
$$PN \cdot QS : OP . OQ;$$

therefore $\quad PM . PN \quad QR : QS,$

and therefore Q is a point on the locus; that is every point on a certain straight line through O satisfies the given condition

Similarly there will be a line OP' dividing the angle at O supplementary to BOA

Secondly, let the lines AB, CD be parallel



Let P be a point on the locus, and let QP be parallel to AB,

then $\quad QA \quad QC . PB \cdot PD,$

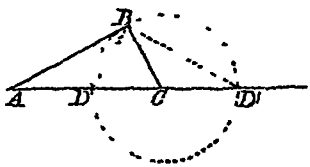
and therefore Q is a point on the locus,

that is the locus consists of the line QP parallel to AB and CD

Similarly there will be a line parallel to AB dividing BD externally in the same ratio.

ii. *The locus of a point whose distances from two fixed points are in a constant ratio (not one of equality) is a circle.*

Let A, C be the fixed points, and let B be one of the points on the locus, such that $AB \quad BC$ in the given constant ratio



Let D, D' divide AC internally and externally in the given ratio, so that

$$AD \cdot DC \quad AD' \quad D'C$$
$$. \ AB : BC$$

Join DB, $D'B$.

Then, since $AB \quad BC \ . \ AD \cdot DC$,

DB bisects the angle ABC, (Th 9)

and since $AB \quad BC \quad AD' \cdot D'C$,

$D'B$ bisects the exterior angle at B, (Th 9)

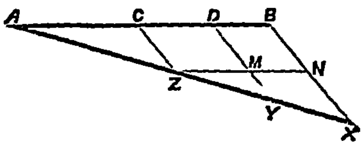
but the bisectors of adjacent supplementary angles are at right angles to one another,

therefore the angle DBD' is a right angle,

therefore the locus of B is the circle described on DD' as diameter.

PROBLEM 1

To divide a straight line similarly to a given divided straight line.



Let AB be the given divided line, and let it be required to divide AX similarly to AB.

Construction Place AX so as to make an angle with AB

Join BX,

and through C, D, the points of division of AB

draw CZ, DY parallel to BX,

then AX is divided similarly to AB

Proof. Through Z draw ZMN parallel to AB to meet DY, BX in M, N.

Then because CZ is parallel to DY,

therefore $AZ : ZY :: AC : CD$;

and because MY is parallel to NX,

therefore $ZY : YX :: ZM : ZN$

$:: CD : DB$;

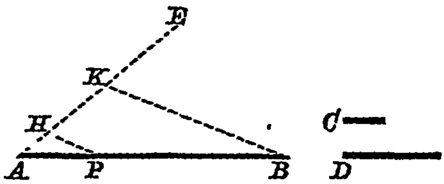
and therefore AX is divided similarly to AB.

PROBLEM 2.

To divide a straight line internally or externally in a given ratio.*

Let AB be the given line, C and D the lines which have the given ratio; then it is required to divide AB into two parts, which have to one another the ratio of $C : D$.

Construction. From A draw a line AE making any angle with AB, and cut off parts AH, HK equal to C and



D respectively. Join KB, and draw HP parallel to KB. P will be the point of division required.

* By a *given* ratio is meant the ratio of two given lines, or of two given numbers. and since two lines can always be found which have the ratio of two given numbers, it follows that a given ratio can always be represented by the ratio of two given lines.

Proof For since HP is parallel to KB,

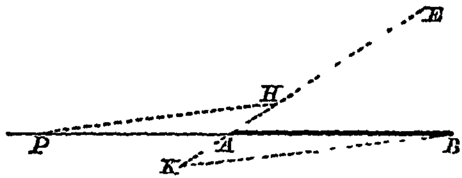
therefore $\quad AP \quad PB \quad AH \quad HK$,

but $\quad AH = C$, and $HK = D$,

therefore $\quad AP \quad PB \quad C \cdot D$,

that is, AB is divided into two parts which are to one another in the given ratio

Note. This construction divides the line *internally* into parts which have the given ratio If it is required to divide



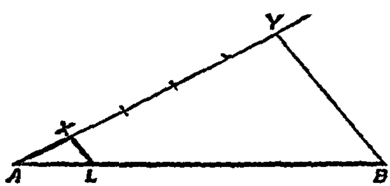
it *externally*, HK must be measured in the opposite direction along AE, as in the figure

The proof will be the same as before.

PROBLEM 3

From a given straight line to cut off any part required

Let AB be the given straight line it is required to cut off from it any part required



Construction Draw any line AX making an angle with AB

produce AX indefinitely; and cut off along it parts equal to

AX until a length AY is obtained which is the same multiple of AX that AB is to be of the part required.

Join BY,

and through X draw XL parallel to YB

Then AL is the part required.

Proof. Because XL is parallel to YB,

therefore $\qquad AL : AB = AX : AY$,

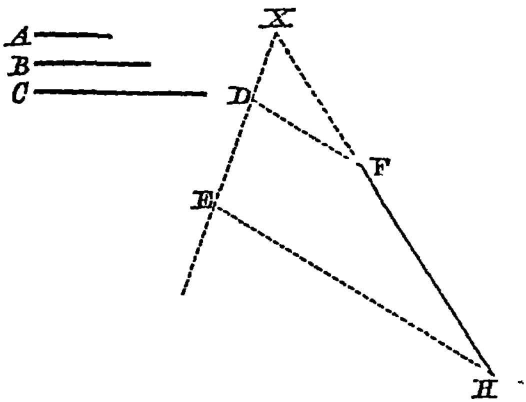
therefore AL is the same part of AB that AX is to AY; that is, AL is the part required

PROBLEM 4.

To find a fourth proportional to three given straight lines

Let A, B, C be the given straight lines to which it is required to find a fourth proportional.

Construction. Take any angle X, and on one of its arms take XD, DE equal to A, B respectively: and on the other arm take XF equal to C Join DF, and draw EH parallel to DF, to meet XF produced in H.



Then shall FH be the line required

Proof For since DF is parallel to EH,

$$XD \quad DE \quad XF \quad FH,$$

but XD, DE, and XF are equal to A, B, C respectively,

therefore $$A \quad B \quad C \quad FH,$$

that is, FH is the fourth proportional required

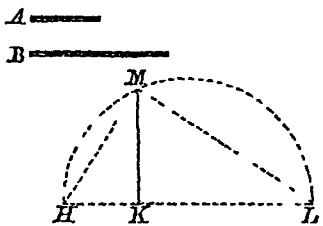
COR. *Hence a third proportional to two given straight lines can be found, by taking $C = B$*

PROBLEM 5

To find a mean proportional between two given straight lines

Let A, B, be the given straight lines it is required to find a mean proportional between A and B

Construction Take HK, KL in the same straight line, equal to A and B respectively On HL describe a semi-



circle, and draw KM perpendicular to HL to meet the circumference in M KM is the line required

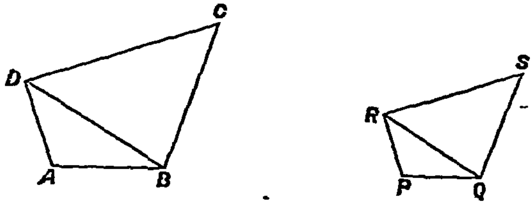
Proof Join HM, ML Then since HML is a semicircle, HML is a right angle, therefore MK, the perpen-

dicular from the right-angle on the hypothenuse, is a mean proportional between the segments of the base; (v. 7 Cor.) that is, MK is a mean proportional between HK and KL, or between A and B

PROBLEM 6.

On a straight line to describe a rectilineal figure similar to a given rectilineal figure.

Let $ABCD$ be the given rectilineal figure, PQ the given straight line: it is required to describe on PQ a rectilineal figure similar to $ABCD$



Construction. Join DB: at P, Q make angles QPR, PQR equal respectively to BAD, ABD. at R, Q, in the straight line RQ make angles QRS, RQS equal respectively to BDC, DBC.

Then will the figure $PQSR$ be similar to the figure $ABCD$.

Proof. By construction the angles of the figure $PQSR$ are respectively equal to the angles of the figure $ABCD$. and by similar triangles ABD, PQR,

$$AB \quad BD \quad PQ : QR,$$

again, by similar triangles DBC, RQS,

$$BD : BC \cdot QR : QS,$$

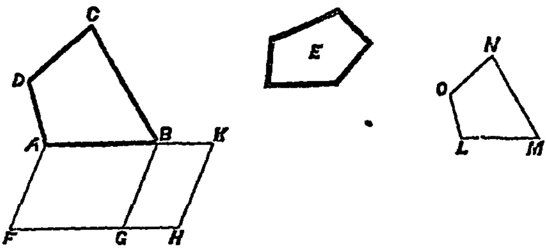
and therefore $\quad AB \,.\, BC \,.\, PQ \,.\, QS.$

In the same manner it may be shewn that the sides about each of the equal angles are proportionals ·
therefore the figure $PQSR$ is similar to the figure $ABCD$

PROBLEM 7

To describe a rectilineal figure equal to one and similar to another given rectilineal figure

Let $ABCD$ be the given rectilineal figure to which the required figure is to be similar, E that to which it is to be equal



Construction On AB construct a parallelogram $AFGB$ equal to $ABCD$, (II Prob 2)

On BG construct a parallelogram equal to E, and having an angle $GBK =$ the angle FAB (II Prob 2)
between AB and BK find a mean proportional LM
(V Prob 5)

On LM describe a rectilineal figure $LMNO$ similar to $ABCD$, so that LM is homologous to AB (V Prob 6)

Proof. Since $AB \,.\, LM : LM \quad BK$,
therefore $AB \,.\, BK$ is the duplicate ratio of $AB \quad LM$.

And since $AB \quad BK \quad AG \,.\, BH$, (IV 4)

and $ABCD \quad LMNO$ is the duplicate ratio of $AB : LM$,

(v. 16 Cor. 1)

therefore $\quad AG \quad BH :: ABCD : LMNO$;

but $\quad AG = ABCD$, (Constr)

therefore the figure $LMNO =$ the parallelogram BH;

but the parallelogram $BH =$ the figure E, (Constr)

therefore the figure $LMNO$ is equal to the figure E,

and it is also similar to the figure $ABCD$.

Miscellaneous Theorems and Problems

1 The bisector of an angle of an equilateral triangle passes through one of the points of trisection of the perpendicular from either of the other angles on the opposite side.

2 The bisectors of the angles of a triangle intersect in one point

3 ABC, PQR are two parallel lines such that

$$AB \quad BC \quad PQ : QR \cdot$$

prove that AP, BQ, CR are either parallel or meet in one point.

4 The external bisector of the vertical angle of an isosceles triangle is parallel to the base.

5. The line joining the middle points of the sides of a triangle is parallel to the base, and is equal to half the base

6. The triangle formed by joining the middle points of the sides of a triangle is similar to the original triangle, and has one fourth of its area.

7. The lines that join the middle points of adjacent sides of a quadrilateral form a parallelogram Under what circumstances will it be a rhombus, a square, or a rectangle?

8 ABC is a triangle, and in AC a point A' is taken, and BB' is cut off from CB produced, so that $AA' = BB'$. Prove that $A'B'$ is cut by AB into parts which have to one another the ratio $CB . CA$

9 To inscribe a square in a triangle

10 If two triangles are on equal bases between the same parallels any straight line parallel to their bases will cut off equivalent areas from the two triangles

11. Make an equilateral triangle equivalent to a given square.

12. Find a point O within the triangle ABC, such that OAB, OAC, OBC shall be equivalent triangles.

13. The angle A of a triangle ABC is bisected by a line that meets the base in D BC is bisected in O. Prove that $$OB . OD . AB+AC . AB \sim AC.$$

14. Given the base, vertical angle, and ratio of the sides, construct the triangle.

15 Perpendiculars are drawn from any point within an equilateral triangle on the three sides, shew that their sum is invariable.

16. Deduce from Ptolemy's Theorem that if P is any point in the circumference of the circle circumscribing an equilateral triangle ABC, of the three lines PA, PB, PC one is equal to the sum of the other two

17. From any point in the base of a triangle lines are drawn parallel to the two sides. Find the locus of the intersection of the diagonals of the parallelograms so formed.

18. In a quadrilateral figure which cannot be inscribed in a circle the rectangle contained by the diagonals is less than the sum of the rectangles contained by the opposite sides.

19 AB is a given line, and CD a given length on a line parallel to AB, and AC, BD intersect in O: prove that as CD varies in position, the locus of O is a line parallel to AB.

20. AB is a diameter of a circle of which AEF, BEG are chords CED is drawn through E at right angles to AB prove that $CFDG$ is a quadrilateral such that the ratio of any pair of its adjacent sides is equal to the ratio of the other pair.

21 Divide a given arc of a circle into two parts which have their chords in a given ratio to one another.

22. If in two similar triangles lines are drawn from two of the equal angles to make equal angles with the homologous sides, these lines shall have to one another the same ratio as the sides of the triangle.

23. To make a rectilineal figure similar to a given rectilineal figure, and having a given ratio to it.

24. To find two straight lines which shall have the same ratio as two given rectangles.

25. To describe on a given straight line a rectangle equal to a given rectangle.

26. To make an isosceles triangle, with a given vertical angle, equal to a given triangle.

27 Let P, Q be points in AB, and AB produced, so that $AP \quad PB \quad AQ \quad QB$, and let O be the middle point of PQ, prove $AO \times BO = OP^2$

28. In any triangle ABC the rectangle $AB \times AC$ is equal to the rectangle contained by the diameter of the circle circumscribing the triangle, and the perpendicular from A on BC

29. Hence shew that if A be the area of a triangle ABC, D the diameter of the circumscribing circle,

$$A \times D = \tfrac{1}{2} AB \times BC \times CA.$$

30 Construct a rectangle equal to a given square, and having the sum of its adjacent sides equal to a given straight line.

31 Construct a rectangle equal to a given square, and having the difference of its adjacent sides equal to a given square.

32 Describe a rectangle equal to a given square, and having its sides in a given ratio.

33 If ABC is a triangle inscribed in a circle, and the tangent at A meets BC produced in D, prove that

$$CD \;.\; BD \;\cdot.\; CA^2 \quad BA^2.$$

34. AB is a diameter of a circle, and at A and B tangents are drawn to the circle If PCQ be a tangent at any point C, cutting the tangents at A, B in P, Q, prove that the radius of the circle is a mean proportional between the segments PC, QC.

35. With the same figure prove that if AQ, BP intersect in R, then CR is parallel to AP or BQ.

36. If two triangles AEF, ABC have a common angle A, prove that

$$\text{triangle } AEF : \text{triangle } ABC = AE \,.\, AF : AB \,.\, AC.$$

37. Given two points in a terminated straight line, find a point in the straight line such that its distances from the extremities of the line are to one another in the same ratio as its distances from the fixed points.

38 Divide a given straight line into two parts such that their squares may have a given ratio to one another.

39 AB is divided in C; shew that the perpendiculars from A, B on any straight line through C have to one another a constant ratio.

40 From the obtuse angle of a triangle to draw a line to the base which shall be a mean proportional between the segments of the base.

41. Divide a given triangle into two parts which shall have to one another a given ratio by a line parallel to one of the sides

42. If from any point in the circumference of a circle perpendiculars be drawn to the sides, or sides produced of an inscribed triangle, prove that the feet of these perpendiculars lie in one straight line.

43. If a line be divided into any two parts to find the locus of the point in which these parts subtend equal angles.

44. If two circles touch each other externally, and also touch a straight line, prove that the part of the line between the points of contact is a mean proportional between the diameters of the circles

45. Any regular polygon inscribed in a circle is a mean proportional between the inscribed and circumscribed regular polygons of half the number of sides

46 ABC is a triangle, and O is the point of intersection of the perpendicular from ABC on the opposite sides of the triangle: the circle which passes through the middle points of OA, OB, OC, will pass through the feet of the perpendiculars, and through the middle points of the sides of the triangle

47 Describe a circle to touch a given straight line and a given circle, and to pass through a given point.

48 A and B are two points on the same side of a straight line which meet AB produced in C Of all the points in this straight line find that at which AB subtends the greatest angle

49 Inscribe a square in a given pentagon

50. $ABCD$ is a quadrilateral figure circumscribing a circle, and through the centre O, a line EOF equally inclined to AB and BC is drawn to meet them in E and F prove that $AE \quad EB \quad CF \quad FD$

CONIC SECTIONS.

CONIC SECTIONS.

BOOK V.

CHAPTER I

ON THE SECTIONS OF A CONE

A *right circular cone* is the solid generated by the revolution of a right-angled triangle round one of the sides containing the right angle

The fixed side is called the *axis* of the cone.

The hypothenuse, which by its motion generates the surface of the solid, is in any position called a *generating line*, which meets the axis in a point called the *vertex*

THE PARABOLA.

Def. The section of a cone made by a plane which is parallel to one of the generating lines of the cone, and perpendicular to the plane which contains that generating line and the axis of the cone, is called a *parabola*

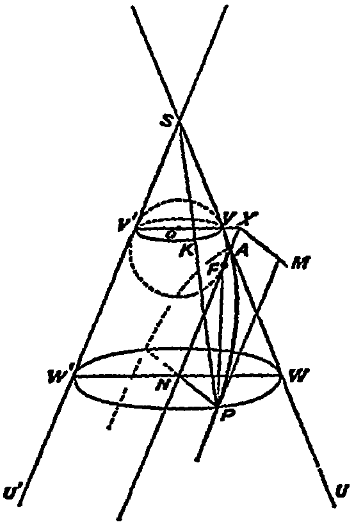
THEOREM I.

In the parabola the distance of every point on the curve from a fixed point in its plane is equal to its distance from a fixed straight line, also in its plane

Let the plane of the paper contain the axis of a right circular cone, and intersect its surface in the generating lines SU, SU', and let a plane, perpendicular to the

plane of the paper, and parallel to SU', intersect the cone in the parabola AP, and the plane of the paper in the line AN.

Let a sphere be described to touch the cone in the circle VKV', and the plane of the parabola in the point F, its centre being in the plane of the paper (Th. 36)



Let the plane of circle VKV' intersect the plane of the parabola in the line XM, which will be perpendicular to the plane of the paper. IV. 18, Cor.

Take any point P on the parabola. Join SP, meeting the circle VKV' in K, join FP, and draw PM perpendicular to XM.

Draw a plane through P perpendicular to the axis of the cone, to cut the cone in the circle WPW', and the plane of the parabola in PN, which will also be perpendicular to the plane of the paper.

Then $FP = KP$, being tangents from P to a sphere

But since $SP = SW$, and $SK = SV$,

$$\therefore KP = VW,$$

and since AN is parallel to SW', by the definition of a parabola, $\therefore$ the angle $ANW = SW'W$

$$= SWW',$$

and therefore ANW is isosceles

So also AVX is isosceles; and therefore $VW = XN$

But $XN = PM$, being opposite sides of a parallelogram, and therefore $FP = PM$

that is, the distance of P, any point on a parabola, from a fixed point F in its plane is equal to its distance from a fixed line XM, also in its plane.

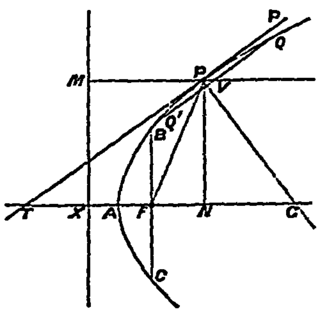
COR *The parabola is symmetrical with respect to the axis* AN.

DEFINITIONS.

The following are definitions of terms used in studying Conic Sections.

The fixed point F is called *the focus.*

The line XM is called *the directrix.*



If FX, perpendicular to XM, meet the curve in A, A is called the *vertex*, and AF produced is called *the axis.*

A straight line PN perpendicular to the axis is called the *ordinate* of P, AN is its *abscissa.*

The double ordinate through the focus is called the *Latus Rectum.*

A line drawn to cut the curve is called a *secant.*

A line drawn to touch the curve at P is called *the tangent* at P; PG perpendicular to the tangent at P, and meeting the axis in G, is called the *normal*

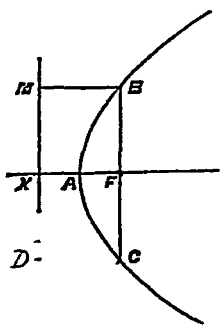
NT is called the *subtangent;* NG the *subnormal.*

A line MPV parallel to the axis of a parabola is called a *diameter*, and a line QV parallel to the tangent at P is called an *ordinate to the diameter through P*, PV is the corresponding *abscissa.* The focal chord parallel to PT is called the *parameter* of the diameter through P

THEOREM 2. THE LATUS RECTUM

The Latus Rectum BC = 4AF.

Let BC be the latus rectum; draw BM perpendicular to the directrix.



Then $BF = BM$, by the property of the parabola,

$$= XF$$

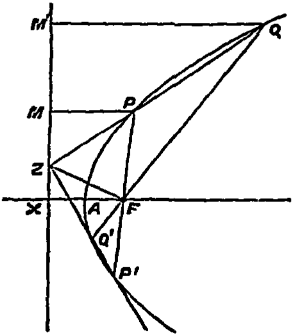
$$= 2AF,$$

since $AF = AX$, by the property of the parabola,

$$\therefore BC = 4AF.$$

THEOREM 3. THE SECANT

If a secant PQ *meets the directrix in* Z, ZF *is the bisector of the exterior angle between the focal distances* FP, FQ



Draw PM, QM' perpendicular to the directrix

Then, by similar triangles ZPM, ZQM',

$$PZ : QZ = PM : QM'$$
$$= PF : QF,$$

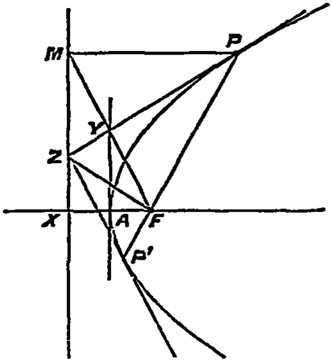
$\therefore$ FZ is the exterior bisector of PFQ Eucl. VI A.

COR 1 *If* PF, QF *produced meet the curve again in* P′, Q′, FZ *is also the bisector of the exterior angle between* P′F, Q′F, *therefore* P′Q′ *passes through* Z

COR 2 PQ′, QP′ *produced intersect on the directrix in some point* Z′, *such that* FZ′ *bisects the angle* Q′FP′ *by Cor.* 1, *and therefore* FZ, FZ′ *are the bisectors of the adjacent angles* PFQ′, Q′FP′; *and therefore* ZFZ′ *is a right angle.*

Theorem 4. The Tangent.

The tangent at P *bisects the angle between the focal distance of* P *and the perpendicular from* P *on the directrix, and* PZ *subtends a right angle at the focus.*



The tangent at P is the limiting position of the secant PQ in the figure of Theorem 3, when Q moves up to P and therefore FQ coincides with FP.

Therefore if PZ is the tangent at P, meeting the directrix at Z, PFZ is a right angle.

Hence in the right-angled triangles PMZ, PFZ, since PZ is common, and $PM = PF$, we have $MPZ = FPZ$

Cor. 1. If PFP' is a focal chord, the tangents at its extremities intersect in the directrix.

For since ZFP is a right angle, ZFP' is also a right angle, therefore ZP' also subtends a right angle at F, and is therefore the tangent at P'.

Cor 2. PZP' is a right angle.

For PZ and $P'Z$ bisect the adjacent angles MZF, XZF Hence *tangents at the extremities of a focal chord intersect at right angles in the directrix.*

COR 3 If FM cuts PZ in Y, it follows from the triangles PMY, PFY that $MY = YF$, and that the angles at Y are right angles.

Join AY, and since $FY = YM$ and $FA = AX$, AY is parallel to the directrix, and is therefore the tangent at A

Therefore *the locus of the foot of the perpendicular from the focus on the tangent is the tangent at the vertex*

COR. 4 Since FYM is perpendicular to the tangent and $FY = YM$, M is called the *image* of the focus in the tangent. It follows that *the locus of the image of the focus in the tangent is the directrix*

THEOREM 5. SEGMENTS OF THE AXIS

If NT *is the subtangent,* NG *the subnormal, to prove*

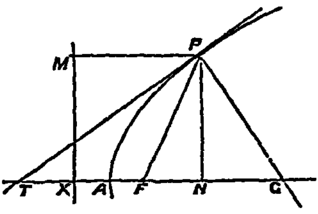
$$NT = 2AN \text{ and } NG = 2AF.$$

Since $$FPT = TPM \text{ (Th 4)},$$

$$\text{and } TPM = \text{the alternate angle } PTF,$$

$$\therefore FPT = PTF,$$

$$\therefore FP = FT$$



And since $FP = PM = XN$,

$$\therefore FT = XN,$$

but $AF = XA$,

$$\therefore AT = AN,$$

and $NT = 2AN.$

Again, since TPG is a right angle, FPG is the complement of FPT, and FGP the complement of FTP;

$$\therefore FGP = FPG \text{ and } FP = FG.$$

$$\therefore FG = FP = PM = XN,$$

and taking away FN,

$$\therefore NG = FX$$
$$= 2AF.$$

THEOREM 6. ORDINATE AND ABSCISSA

The square of the ordinate is equal to the rectangle contained by the abscissa and the latus rectum.

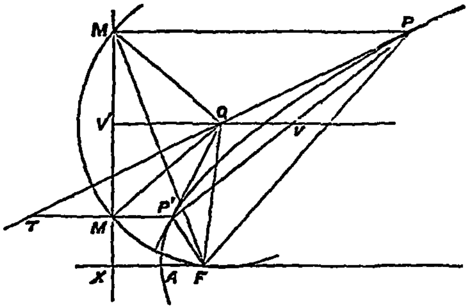
$$(PN^2 = 4AF \, . \, AN)$$

Referring to the last figure, since the angle TPG is a right angle,

$$PN^2 = TN \, . \, NG$$
$$= 2AN \times 2AF \text{ (by Theorem 5)}$$
$$= 4AF \, . \, AN.$$

THEOREM 7 PAIRS OF TANGENTS

Tangents from any point subtend equal angles at the focus, and have equal projections on the directrix, and the triangles formed by the tangents with the focal distances are similar.



Let QP, QP' be tangents drawn from Q, PM, $P'M$ perpendiculars to the directrix

Then by the equal triangles FPQ, MPQ, $FQ = MQ$, and $QMP = QFP$.

Similarly $M'Q = FQ$, and $QM'P' = QFP'$

∴ Q is the centre of a circle $MM'F$, and the chord MM' is the projection of PP' on the directrix.

And since $QM = QM'$, $QMM' = QM'M$,

∴ the angles QMP, $QM'P'$ are equal

But since $QMP = QFP$, and $QM'P' = QFP'$,

∴ $QFP = QFP'$,

that is, *tangents subtend equal angles at the focus.*

Again, since QM, QM' are equal, and equally inclined to MM', the diameter through Q will bisect MM', and therefore the projections MV', $M'V'$ of QP, QP' on the directrix are equal.

Again, by joining FM, since FMM', QPM are each complementary to FMP;

$$\therefore FMM' = QPM,$$

$\therefore$ $FPQ = QPM = FMM' = \frac{1}{2} FQM'$ (which is the angle at the centre Q on the same arc FM')

$$= FQP'.$$

Hence the triangles QPF, $P'QF$ are similar.

Cor. 1. The diameter through Q bisects PP' in V. For PV, $P'V'$ have equal projections MV', $M'V'$ on the directrix.

Cor. 2. If PQ meet $P'M'$ in T;

since $FQP' = FPQ = QPM = QTP'$

and $FP'Q = QP'T$, (Th. 4),

therefore *the triangles* FP'Q, QP'T *are similar*.

Cor. 3. Hence *a pair of tangents can be drawn to a parabola from any external point.*

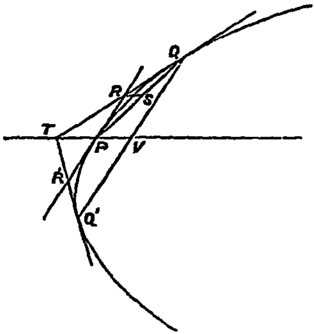
Let Q be the given point, describe a circle with centre Q, and radius QF, to meet the directrix in M, M', and draw MP, $M'P'$ perpendicular to the directrix to meet the curve in P, P'. Then QP, QP' will be the tangents. For from the equal triangles FPQ, MPQ, the angle FPQ = the angle MPQ, and therefore QP is the tangent at P (Th. 4).

THEOREM 8 DIAMETERS

A diameter bisects all chords parallel to the tangent at its extremity

Let PV be a diameter, PR the tangent at P meeting the tangent at Q in R, and let QQ' be parallel to PR

Then will QQ' be bisected in V



Let the tangent at Q meet PR in R

Draw RS parallel to the axis

Then $QS = SP$ (by Th 7, Cor 1), and $\quad TR = RQ$,

and $\quad TP = PV$.

Similarly if the tangent at Q' meet VP produced in T', $T'P = PV$, $\therefore$ T and T' are identical, that is, the tangents at QQ' intersect on the diameter through P

But the diameter through T bisects QQ' (Th 7),

$\therefore$ the diameter through P bisects all chords parallel to the tangent at P.

COR. $QV = 2PR$, for $QV . RP . TV \quad TP$

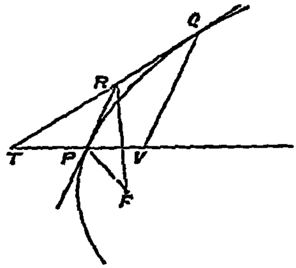
THEOREM 9. OBLIQUE ORDINATE AND ABSCISSÆ.

If QV is the ordinate to the diameter PV,

$$QV^2 = 4FP \cdot PV.$$

For $QV = 2PR$;

and $\therefore QV^2 = 4PR^2$:



let QR meet PV in T, then the triangles FPR, RPT are similar, by Th. 7, Cor. 2,

$\therefore FP : PR :: PR : TP$;

$\therefore PR^2 = FP \cdot TP$;

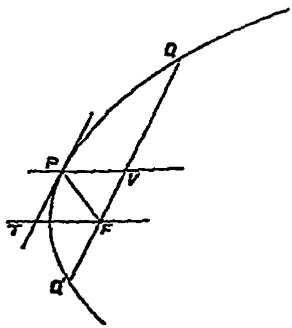
$\therefore QV^2 = 4FP \cdot PT$;

but $PT = PV$ (Th. 8),

$\therefore QV^2 = 4FP \cdot PV$.

THEOREM 10. THE PARAMETER.

The parameter of the diameter through $P = 4FP$



Let $QVFQ'$ be parallel to PT, the tangent at P;

then $FP = FT = PV$ (Th. 5),

but $QV^2 = 4FP \cdot PV$ (Th. 9)

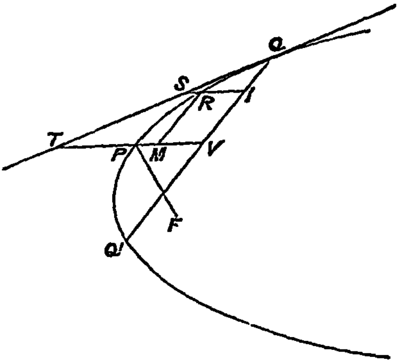
$= 4FP^2$,

$\therefore QV = 2FP$,

and $\therefore QQ' = 4FP$.

THEOREM 11. SEGMENTS OF DIAMETER MADE BY TANGENT AND CHORD.

If a diameter of a parabola is cut by a chord, and the tangent at the extremity of the chord, the segments of the diameter made by the curve are in the same ratio as the segments of the chord



Let the diameter SRI meet the chord QQ' in I, the curve in R, and the tangent QT in S, then is

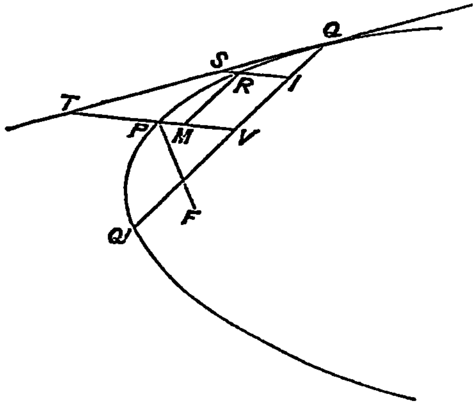
$$SR \,.\, RI \;:\; QI \,.\, IQ'.$$

Draw TPV the diameter to which QQ' is an ordinate, and draw RM parallel to QQ', to meet PV in M Join PF

Then because $\quad QV^2 = 4FP \quad PV \qquad$ (Th 9)

and $\quad RM^2 = 4FP \,.\, PM, \qquad$ (Th 9)

$$QV^2 - RM^2 = 4FP \,.\, MV,$$



but
$$QV^2 - RM^2 = QV^2 - IV^2$$
$$= QI \, . \, IQ',$$
and
$$MV = RI;$$
$$\therefore \; QI \, . \, IQ' = 4FP \, . \, RI,$$
and
$$\therefore \; RI : IQ' :: IQ : 4FP. \qquad (1)$$

Again, by similar triangles
$$SI : IQ :: TV : QV,$$
$$:: 2PV : QV, \qquad \text{(Th 8)}$$
$$:: QV : 2FP,$$
(because $QV^2 = 4FP \, . \, PV$)
$$:: QQ' : 4FP,$$
and
$$\therefore \; SI : QQ' :: IQ : 4FP. \qquad (2)$$

Therefore combining (1) and (2)
$$RI : IQ' :: SI : QQ'$$
and
$$RI : SI :: IQ' : QQ',$$
or
$$SR : RI :: QI : IQ'.$$

COR. $QI^2 = 4FP \cdot SR$

For $$SR \quad RI \cdot QI \quad IQ',$$

$$\therefore 4FP \cdot SR \quad 4FP \quad RI \quad QI^2 \quad QI \quad IQ,$$

but $$4FP \quad RI = QI \cdot IQ',$$

$$QI^2 = 4FP \cdot SR.$$

This property of the parabola is of great use in the theory of projectiles

THEOREM 12. SEGMENTS OF INTERSECTING CHORDS

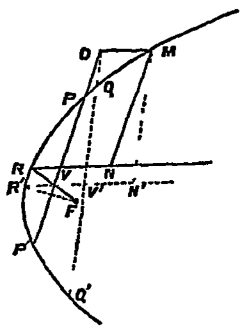
If two chords of a parabola PP′, QQ′ *intersect in* O, PO OP′ QO OQ′ *in the ratio of the parameters of the diameters which bisect the chords*

Draw the diameter RV bisecting PP', draw OM parallel to the axis, and MN parallel to OV

Then $PO \quad OP' = OV^2 - PV^2$

$$= MN^2 - PV^2$$

$$= 4FR \quad RN - 4FR \quad RV$$



by Th 9,

$$= 4FR \quad VN$$

$$= 4FR \quad OM$$

Similarly $QO \quad OQ' = 4FR' \quad OM,$

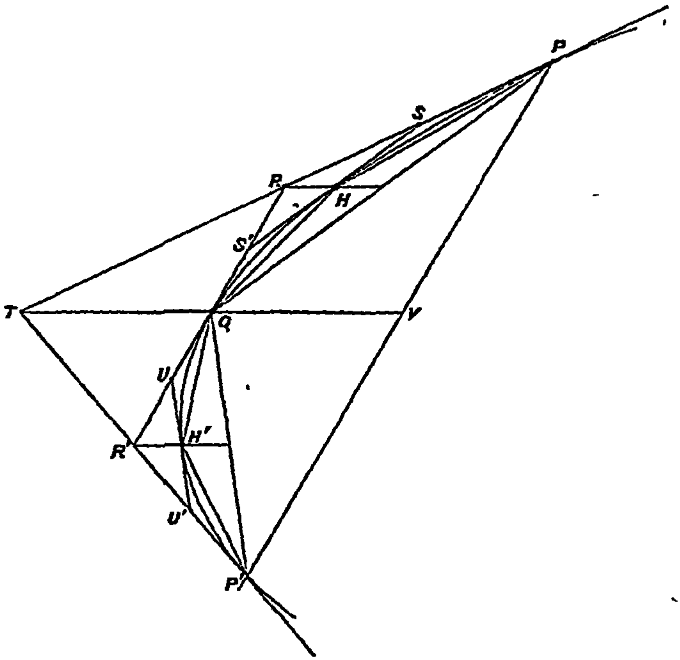
$$PO \quad OP' \quad QO \quad OQ' \quad 4FR \cdot 4FR',$$

which proves the proposition

THEOREM 13. AREA OF PARABOLA.

The area of a parabola cut off by any chord is two thirds of the area of the triangle formed by the chord and the tangents to the parabola at its extremities

Let PP' be any chord of the parabola PQP', PT, $P'T$ the tangents to the parabola at the extremities of PP', then will the area included between the curve PQP' and the chord PP' be $\frac{2}{3}$ of the area of the triangle PTP'.



Draw TQV parallel to the axis to meet the curve in Q and the chord in V, and draw the tangent RQR'. Join QP, QP'.

Then, since $TR = \frac{1}{2} TP$, (Th 8),

the triangle $TQR = \frac{1}{2}$ of the triangle TQP,

and, since $TQ = QV$, (Th 8),

the triangle $TQP =$ the triangle PQV,

. the triangle $TQR = \frac{1}{2}$ of the triangle PQV

Similarly the triangle $TQR' = \frac{1}{2}$ of the triangle $P'QV$,

. the triangle $RTR' = \frac{1}{2}$ of the triangle PQP'

Again, by drawing diameters through R, R', to meet the parabola in H, H', and drawing tangents SS' at H, and UU' at H', and joining PH, HQ, QH', $H'P'$, it may be similarly proved that the triangles SRS', $UR'U'$ are respectively halves of the triangles PHQ, $QH'P'$.

And therefore, by adding, the area $TSS'UU'T$ is half the area $PHQH'P'P$

And by continuing this process, drawing diameters through S, S', U, U', and drawing tangents at the points where these diameters meet the curve, it is plain that the polygon formed by the tangents outside the curve is *always* half the polygon formed by the chords inside the curve

And therefore this is true when the number of the sides of the polygon is indefinitely increased

But in the limit the exterior polygon becomes the area included by the tangents and the curve, and the interior polygon becomes the area included by the chord and the curve,

therefore the exterior area $= \frac{1}{2}$ the interior area;

and therefore the interior area $= \frac{2}{3}$ of the whole area,

$= \frac{2}{3}$ of the triangle PTP' [1]

EXERCISES ON THE PARABOLA.

1. If FY is the perpendicular from the focus F to the tangent at P, prove that $FY^2 = AF \cdot FP$.

2. If QP, QP' are two tangents to a parabola, F the focus, prove that

$$QF^2 = PF \cdot P'F.$$

3. The tangent at any point cuts the directrix and the latus rectum produced at points equally distant from the focus

4. To construct a parabola having given two points on the curve, and either the focus or the directrix.

5. To construct a parabola having given the focus, one point, and either one point on the directrix, or one tangent.

6. If Q be any point on the tangent at P, QR, QL perpendicular to the directrix and FP respectively, prove that $QR = FL$.

[1] This theorem is due to Archimedes. It was the first instance of the quadrature of a curvilinear area; that is, of finding a rectilineal area (which can be converted into a square) exactly equal to a curvilinear area

7 The focal distance of a point is greater than, equal to, or less than its distance from the directrix according as the point is outside, on, or inside the parabola

8 If PM, $P'M'$ are perpendiculars on the directrix from the extremities of a focal chord PFP', prove that MFM' is a right angle.

9. PN, $P'N'$ are the ordinates of the extremities of a focal chord, prove that $PN \times P'N' = \left(\frac{1}{2} \text{ lat rect.}\right)^2$.

10 Hence prove that $FN \times FN' = XZ^2$.

11 Given two tangents at right angles to one another, and their points of contact, to find the vertex.

12 The chord of contact of two tangents from Q subtends the same angle at the focus, that its projection on the directrix subtends at Q.

13. If a parabola touches three sides of a triangle, its focus will lie on the circle circumscribing the triangle

14 If QP, QP' are tangents from Q, prove that

$$QP^2 : QP'^2 :: FP : FP'.$$

15. Prove that the lengths of two tangents from any point are as the perpendiculars on them from the focus

16 Prove that $PG^2 \propto FP$

17 If $FP \,.\, FP'$ is constant, prove that the locus of the intersection of the tangents at P, P' is a circle.

18 Prove that the circle on FP as diameter touches the tangent at the vertex.

19 Prove that the circle on any focal chord as diameter touches the directrix.

20. A point moves so that its distance from a circle is equal to its distance from a diameter of that circle. Shew that it moves in a parabola.

21. Prove that normals at the extremities of a focal chord intersect on the diameter which bisects the chord.

22. Find the focus and directrix of a parabola that touches four straight lines.

23. If two tangents to a parabola be cut by a third the alternate segments will be proportional.

24. Find the locus of points, such that the sum or difference of their distances from a fixed point or circle and a fixed straight line are given.

25. If a parabola roll on an equal parabola, their vertices having been placed together, the focus of the former will describe the directrix of the latter.

26. As the latus rectum is to the sum of any two ordinates, so is the difference of these ordinates to the difference of the abscissæ. (Th 6)

27. Prove Theorem 6 directly from the figure in Th 1

28. Any secant through the focus is harmonically divided by the focus and directrix.

29 If from the point of contact of a tangent with a parabola two lines be drawn to the vertices of any two diameters, each to intersect the other diameter, then the line joining these two points of intersection will be parallel to the tangent. (Th 11)

CHAPTER II.

THE ELLIPSE AND HYPERBOLA PROPERTIES COMMON TO BOTH CURVES.

THE ellipse and hyperbola are *central conic sections*, that is they have a centre, in which, as will appear, every chord that passes through it is bisected. The Parabola has no centre. Hence the ellipse and hyperbola may be conveniently studied together, many of their properties being identical.

In the present chapter the proofs of the properties common to the ellipse and hyperbola are given, with figures of both curves

In the next chapter some properties are given which are either different for the two curves, or are most easily obtained by different modes of proof.

THEOREM I.

An Ellipse has the following properties:

(1) *There are two points in its plane such that the sum of their distances from any point on the curve is constant*

(2) *The ratio of the distances of every point on the curve from a fixed point and fixed straight line in its plane is constant*

(3) *There exists a line in the plane of the ellipse such that the ordinates of the ellipse to abscissæ measured along this line are to the ordinates of the circle described on this line as diameter in a constant ratio.*

A Hyperbola has the following properties:

(1) *There are two points in its plane such that the difference of their distances from every point on the curve is constant*[1].

(2) *The ratio of the distances of every point on the curve from a fixed point, and a fixed straight line in its plane, is constant.*

(3) *There exists a line in the plane of the hyperbola such that the ordinates of the hyperbola to abscissæ measured along this line produced, are to tangents drawn from the feet of these ordinates to the circle described on this line as diameter in a constant ratio.* (Vid. fig of Th. 7.)

See figure at the end of the book.

Let S be the vertex of a right circular cone of which SOO' is the axis, and let the plane of the paper contain the axis and the generators SVU, $SV'U'$; and let any plane perpendicular to the plane of the paper, and intersecting it in AA', obliquely to the axis, cut the surface in the ellipse or the hyperbola APA'. (Vid. p 45.)

Since the plane of the paper is perpendicular to the plane APA', the centres of the spheres which touch the

[1] The proof of (1) is the same as that in the corresponding theorem on the ellipse, the sum of the distances being changed into their difference.

The proof of (2) is also the same as in the ellipse.

The proof of (3) is also the same, the ordinate from N to the circle being changed into the tangent from N to the same circle

plane APA' along the line AA' will be in the plane of the paper. (IV 35, Cor 3)

Hence, if O, O' are the centres of circles which touch AA' and the generators SA, SA' (that is, centres of the inscribed and escribed circles of the triangle SAA'), spheres may be described with centres O, O' to touch the plane APA' in two points F, F' on the line AA', and to touch the cone along two circles whose planes are perpendicular to the axis, that is, along VKV', and $UK'U'$. (IV 36)

Let P be any point on the ellipse, $SKPK'$ the generator passing through P, touching the spheres in K, K'.

Join FP, $F'P$.

Then (1) in the ellipse $FP+F'P=$ a constant.

For $FP=KP$, being tangents to a sphere whose centre is O from the same point P. (IV 36, Cor)

And $F'P=K'P$ for a similar reason.

Therefore

$$FP+F'P=KP+K'P=KK'=SK'-SK=VU,$$

which is constant for all positions of P.

The points F, F' are called the *foci*

It follows from well-known theorems in plane geometry on the inscribed and escribed circles that

$$VU=AA', \text{ and that } AF=A'F', \text{ and } FF'=SA'-SA.$$

(2) Let the plane of the circle $V'KV$ intersect the plane APA' in the line XM, which will therefore be at right angles to the plane of the paper. (IV. 18, Cor)

From P draw PM perpendicular to XM.

Then shall PF be to PM in a constant ratio.

Draw a plane through P perpendicular to the axis of the cone, intersecting APA' in PN, which will therefore be at right angles to AA' (IV. 18, Cor), and meeting the cone in the circle WPW'. (Vid p 44)

Then $PF = PK = VW$,

and $PM = NX$,

$\therefore PF . PM :: VW . NX$,

$:: VA : AX$, since XV is parallel to NW,

$: AF : AX$;

that is, $PF : PM$ in a constant ratio for all positions of P.

Similarly, if the plane $UK'U'$ intersect APA' in $X'M'$, and PM' is drawn perpendicular to $X'M'$,

$PF' : PM' : W'U' . NX'$,

$\cdot\cdot A'U' : A'X'$,

$:: A'F' : A'X'$.

The lines XM, $X'M'$ are called directrices, and F, F' the corresponding foci.

It must be observed that

$AF : AX \cdot AV : AX \cdot VU . XX'$,

$:: V'U' : XX' :: A'U' : A'X' :: A'F' : A'X'$.

The ratio $PF \cdot PM$ is called the eccentricity of the ellipse, and is generally denoted by the letter e.

e is less than 1 in the ellipse, since AV is then less than AX; and e is greater than 1 in the Hyperbola, since AV is then greater than AX.

Also, since $AF = A'F'$, therefore also $AX = A'X'$.

(3) If WPW' is the circular section through P, draw AB, $A'B'$ parallel to VV'

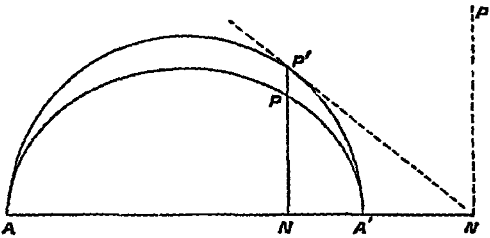
Then $$PN^2 = WN \quad W'N,$$

but $$WN \quad AN \cdot A'B' \quad AA',$$

and $$W'N \quad A'N \quad AB \quad AA',$$

$$WN \,.\, W'N \,.\, AN \quad A'N \quad AB \times A'B' \quad AA'^2,$$

PN^2 is to $AN \quad A'N$ in a constant ratio



But if on AA' as diameter a circle were described, and $P'N$ were the ordinate to it through N, $P'N^2 = AN \times A'N$

Therefore $PN^2 \,.\, P'N^2$, or $PN \quad P'N$, is a constant ratio

The ellipse therefore has this property, that its ordinate bears a constant ratio to the corresponding ordinate of a circle described on AA' as diameter

This circle is called the *auxiliary* circle, and the points P, P' are called *corresponding* points

Both curves are from their mode of construction symmetrical with respect to AA', and since they may be described from either focus and directrix, they must also be symmetrical with respect to an axis bisecting AA' at right angles

Hence every chord through the intersection of the axes will be bisected in that point, and is called a *diameter.*

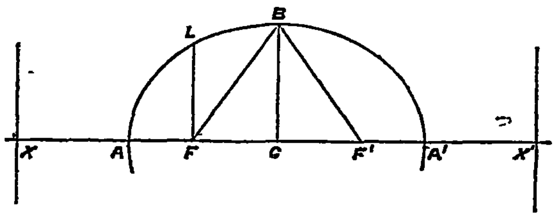
The ellipse will be a closed curve, and the hyperbola will consist of two infinite branches, as may be seen in the figures of Theorem 2.

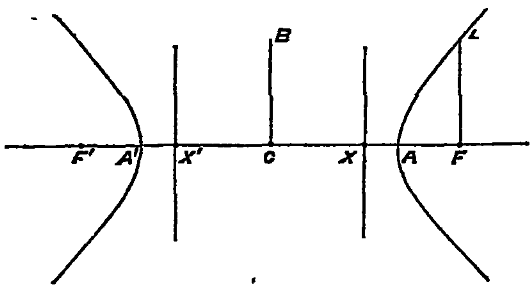
AA' is called the *major* axis or *transverse axis.*

THEOREM 2. SEGMENTS OF THE AXIS.

If A, A' *are the vertices of a central conic,* F, F' *the foci,* X, X' *the feet of the directrices,* C *the middle point of* FF', *then*

$$AF : AX \cdot CF : CA \cdot CA \quad CX$$





For $$AF : AX \,.\, A'F : A'X,$$

by property (2) of the central conic;

$$\therefore AF : A'F :: AX \,.\, A'X;$$

$$\therefore AF : FF' :: AX : AA',$$

$$\therefore AF : AX :: FF' : AA'$$

$$:: CF : CA.$$

For a similar reason, $AF : AA' :: AX : XX'$,

$$\therefore AF : AX :: AA' : XX'$$

$$:: CA : CX$$

COR. 1. $CF \cdot CX = CA^2$.

COR 2 If CB is drawn at right angles to AA' to meet the ellipse in B, then $BC^2 = AF . A'F$.

For in the ellipse, since $FB = F'B$, and

$$FB + F'B = AA', \therefore FB = AC,$$

$$\therefore AF . A'F = FB^2 - FC^2,$$

$$= BC^2.$$

In the hyperbola a line BC at right angles to AA' through its middle point is taken, such that

$$BC^2 = AF \cdot A'F,$$

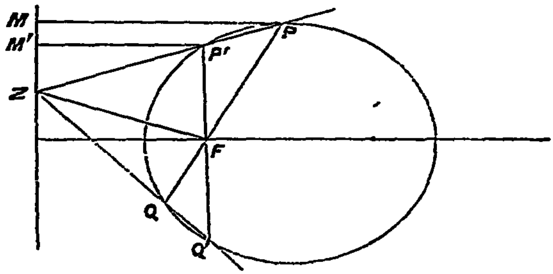
$$\text{or} \quad = FC^2 - AC^2,$$

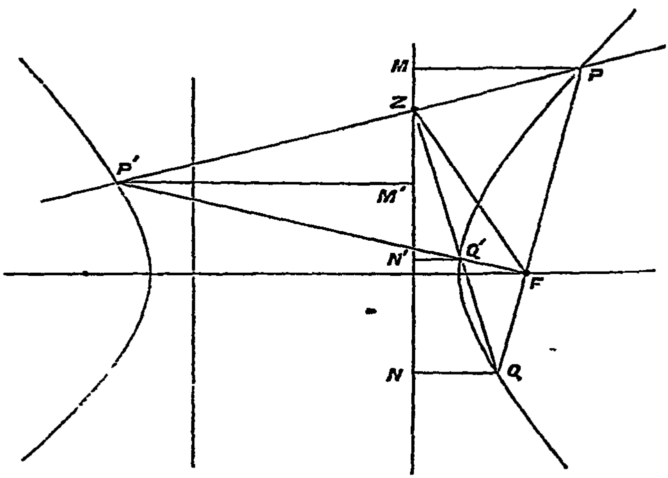
and $2BC$ is called the *minor axis*, or the *conjugate axis* of either ellipse or hyperbola.

THEOREM 3 THE SECANT AND DIRECTRIX.

If in a central conic a secant PP′ *meet the directrix in* Z, *and* F *is the corresponding focus,* FZ *is the exterior bisector of the angle* PFP′, *or of its supplement*

Draw PM, $P'M'$ perpendicular to the directrix.





Then $FP : PM :: FP' : P'M'$,

by a property of a central conic; (Th 1.)

$$\therefore FP : FP' :: PM : P'M',$$
$$:: PZ : P'Z,$$

by similar triangles; therefore FZ bisects the exterior or interior angle of the triangle PFP'.

It will be observed that in the hyperbola FZ is the bisector of the exterior or interior angle of the triangle PFP', according as the secant meets one branch only or both branches of the curve.

COR. 1 *If* PQ, P′Q′ *be focal chords,* QQ′ *and* PP′ *intersect on the directrix*

For PP', QQ' both meet the directrix where it is cut by the bisector of $P'FQ$

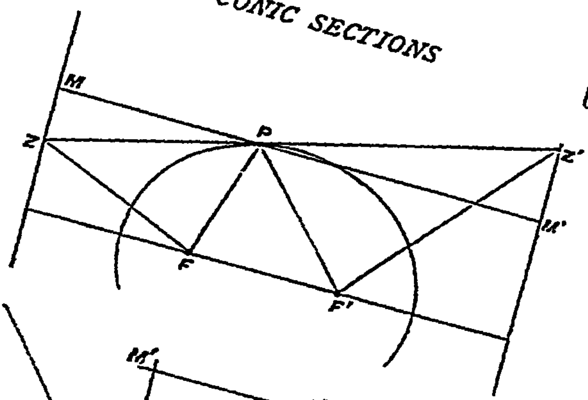
COR. 2 P′Q, PQ′ *also intersect on the directrix in a point* Z′ *by* Cor 1, *and* ZFZ′ *is a right angle, since* FZ *and* FZ′ *are bisectors of adjacent supplementary angles*

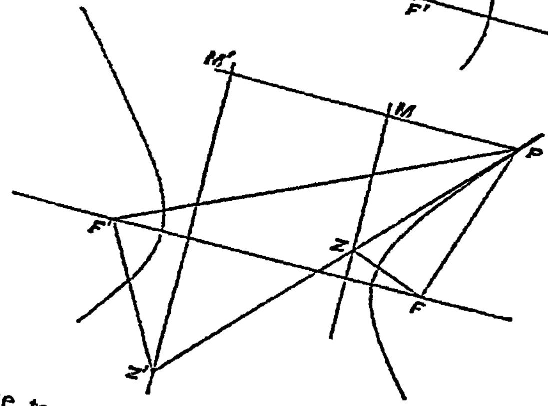
COR. 3 *The tangent being the limiting position of the secant, when the points of intersection approach each other, it follows that when the secants* PP′, QQ′ *become tangents, the tangents at the extremities of a focal chord intersect in the directrix and subtend right angles at the focus.*

THEOREM 4 THE TANGENT IN A CENTRAL CONIC.

The tangent in a central conic makes equal angles with the focal distances

Let ZPZ' be the tangent at P, meeting the directrices in Z, Z'





Since the tangent at P is the limiting position of the secant PP' when P' moves up to P,

$$FZ \text{ is at right angles to } FP. \quad \text{(Th 3, Cor. 3)}$$

Similarly $F'Z'$ is at right angles to $F'P$

And $\therefore FP : F'P :: PM : PM'$, (Th 1)

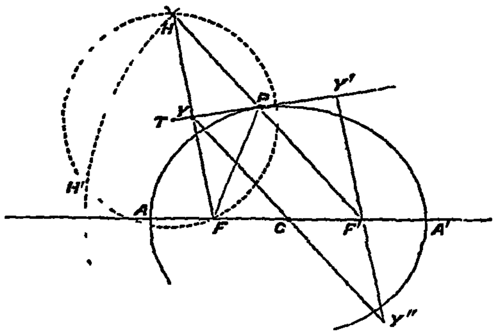
and $PM : PM' :: PZ : PZ'$ by similar triangles,

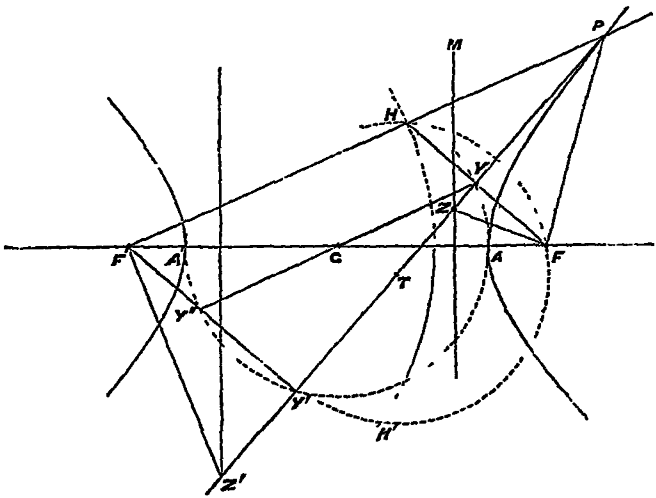
$$\therefore FP : PZ :: F'P : PZ'.$$

Therefore the right-angled triangles PFZ, $PF'Z'$ have the sides about one of the other angles proportionals; therefore they are similar; and therefore

$$FPZ = F'PZ'.$$

COR. I *If* Y *is the foot of the perpendicular from the focus on the tangent,* H *the image of the focus in the tangent, the loci of* Y, H *are circles*





Since $$FY = YH,$$ (p 99)

and the angles at Y are right angles;

$$\therefore\ FP = HP, \text{ and } FPY = HPY,$$

but $FPY = F'PY'$ by the theorem;

$\therefore HPY = F'PY'$ and HPF' or $F'HP$ is a straight line; but $F'H = F'P \pm PH$, the upper sign being taken for the ellipse, and the lower for the hyperbola;

$$= F'P \pm FP$$

$$= AA' = \text{constant}.$$

Therefore the locus of H is a circle described round F' as centre with radius equal AA'.

This is called a *director* circle

Since the tangent bisects HF at right angles it follows that if T be any point on the tangent,

$$TH = TF.$$

Again, to find the locus of Y, join YC

Then, since $\quad FY = YH$ and $FC = CF'$;

$$\therefore FY : YH :: FC : CF',$$

and $\therefore$ YC is parallel to $F'H$, and $= \frac{1}{2} F'H$

$$= CA,$$

therefore the locus of Y is the *auxiliary* circle (Th 1)

COR. 2. *Hence a tangent may be drawn to the conic from any point.*

Draw a circle with centre T and radius TF, to cut the director circle whose centre is F' in H, H', and join HF, cutting the curve in P, and join TP. TP is a tangent. For, since $F'H = AA' = F'P \pm FP$, $\therefore$ in the triangles FPT, HPT, the three sides are respectively equal, and $\therefore$ TP makes equal angles at P with the focal distances of

P, and TP is the tangent at P The other tangent is similarly found by joining $H'F'$

COR. 3 *If* F'Y' *is the perpendicular from the other focus on the tangent,*

$$FY \,.\, F'Y' = AF \ \ A'F = BC^2$$

Produce YC to meet $Y'F'$ in Y'', then, since Y' is a right angle, YCY'' is a diameter of the auxiliary circle · and $CY'' = CY$, and $\therefore$ from the triangles CFY, $CF'Y''$, $F'Y'' = FY$,

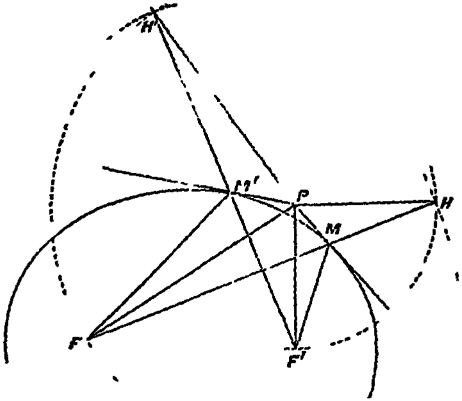
(Euc III 35)

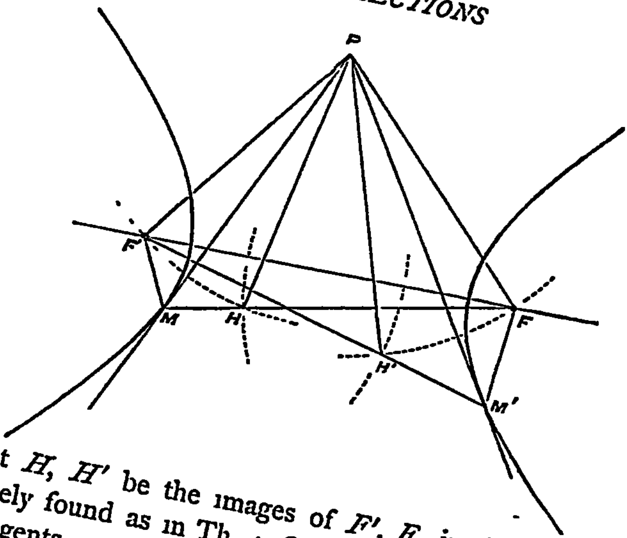
$$\therefore\ FY \ \ F'Y' = F'Y'' \ \ F'Y' = F'A \ \ F'A' = BC^2$$

(Th 2, Cor 2)

THEOREM 5 PAIR OF TANGENTS

The tangents from P *to a central conic make equal angles with the focal distances of* P, *and subtend equal or supplementary angles at either focus*





Let H, H' be the images of F', F, in PM, PM' respectively found as in Th. 4, Cor. 2, and let PM, PM' be the tangents.

Then $FH = AA' = F'H'$, and $PH = PF'$, $PH' = PF$ by Th 4.

Therefore the triangles FPH, $F'PH'$ are equal in all respects;

$$\therefore \text{ the angle } FPH = F'PH',$$

and

$$\therefore \; HPF' = H'PF;$$

$$\therefore \; F'PM = FPM'; \text{ (Halves of equals are equal)}$$

that is, the tangents make equal angles with the focal distances.

Also $PF'M = PHM$; and PHM is equal or supplementary to PHF, that is to $PF'M'$: or the tangents subtend equal or supplementary angles at the focus.

Cor. *If the tangents from* P *include a right angle, the locus of* P *is a circle.*

For if MPM' is a right angle, so is also FPH, since $MPH = M'PF$,

$$\therefore\ FP^2 + F'P^2 = FP^2 + PH^2 = FH^2 = \text{const}$$

But $$FP^2 + F'P^2 = 2CP^2 + 2CF^2,$$

and since $$2CP^2 = FH^2 - 2CF^2,$$ (apply Euc. II-10)

$$= 4AC^2 - 2CF^2,$$

and $$CF^2 = AC^2 \mp BC^2 \qquad \text{(Th 2, Cor 2)},$$

$$\therefore\ CP^2 = AC^2 \pm BC^2,$$

and the locus of P is a circle

THEOREM 6 THE SUBTANGENT ON THE TRANSVERSE AXIS

In a central conic if the tangent at P *meet the transverse axis in* T,

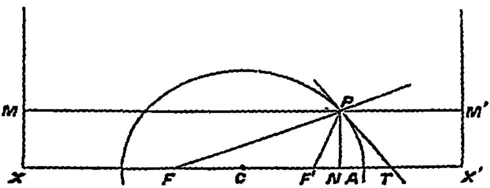
$$\mathrm{CT} \cdot \mathrm{CN} = \mathrm{CA}^2$$

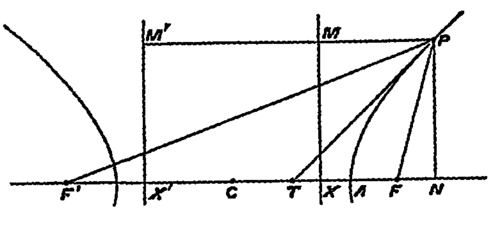
Since PT bisects the angle at P between the focal distances,

$$FT \quad F'T \quad FP : F'P$$

$$PM \cdot PM'$$

$$: XN \cdot X'N,$$





$$\therefore FT+F'T : FT \sim F'T :: XN+X'N : XN \sim X'N,$$
$$\text{or } 2CT : 2CF :: 2CX : 2CN,$$
$$\therefore CT \,.\, CN = CF \,.\, CX$$
$$= CA^2. \qquad \text{(Th 2)}$$

COR. $TA : TN :: TC : TA'$.

CORRESPONDING POINTS AND LINES; THE AUXILIARY CIRCLE.

In the ellipse it was shewn, Theorem 1, that the ordinates to the axis are all less than the ordinates to the same abscissa of the auxiliary circle in the same ratio; i e $PN : P'N$ in a constant ratio.

The points P, P' are *corresponding points*; PQ, $P'Q'$ are called *corresponding lines*

If $B'BC$ is drawn an ordinate through C,

$$PN : P'N :: BC : B'C$$
$$:: BC : AC,$$

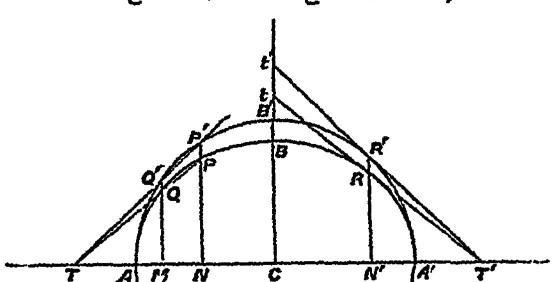
where BC is the semi-axis minor, and AC the semi-axis major.

LEMMA *Corresponding lines in the ellipse intersect on the axis.*

Let PQ, $P'Q'$ be corresponding lines, and let PQ meet the axis in T. Then T is determined by the ratio

$$MT : NT :: QM : PN,$$

but $\qquad QM : PN :: Q'M : P'N,$



and therefore the point where $P'Q'$ meets the axis is determined by the same ratio Therefore PQ, $P'Q'$ intersect on the axis

COR. *Tangents at corresponding points intersect on the axis.*

THEOREM 7 ORDINATE AND ABSCISSA

In a central conic

$$PN^2 . AN \quad A'N \; . BC^2 . AC^2$$

(1) In the ellipse (using the figure in the Lemma),

Let P, P' be corresponding points, then

$$PN^2 . P'N^2 \quad BC^2 \quad AC^2$$

But
$$P'N^2 = AN \quad A'N,$$

$$\therefore PN^2 \cdot AN \quad A'N \quad BC^2 \quad AC^2$$

(2) In the hyperbola.

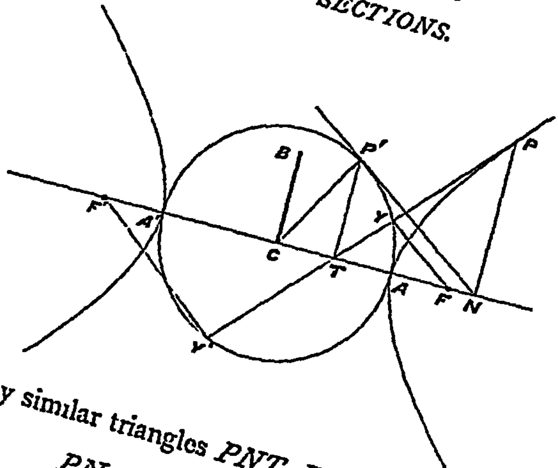
Let PN be the ordinate, NP' the tangent to the auxiliary circle from N, and let the tangent from P meet the axis in T

Then, since

$$CT \quad CN = CA^2 = CP'^2, \qquad \text{(Th 6)}$$

T is the foot of the perpendicular from P' on the axis

Draw FY, $F'Y'$ perpendicular to the tangent



Then, by similar triangles PNT, FYT,

$$PN : FY :: TN : TY,$$

and similarly, $$PN : F'Y' :: TN : TY';$$

$$\therefore PN^2 : FY \cdot F'Y' :: TN^2 : TY \cdot TY',$$

$$\therefore PN^2 : BC^2 :: TN^2 : P'T^2 \text{ (Th. 4, Cor 3)}$$

$$:: P'N^2 : CP'^2$$

$$:: AN \cdot A'N : AC^2;$$

$$\therefore PN^2 : AN \cdot A'N :: BC^2 : AC^2.$$

Cor. *The latus rectum in a central conic is a third proportional to the axes major and minor.*

For (using the figure in Theorem 2),

$$LF^2 : AF \cdot A'F :: BC^2 : AC^2$$

by the Theorem just proved;

$$\therefore LF^2 : BC^2 :: BC^2 : AC^2,$$

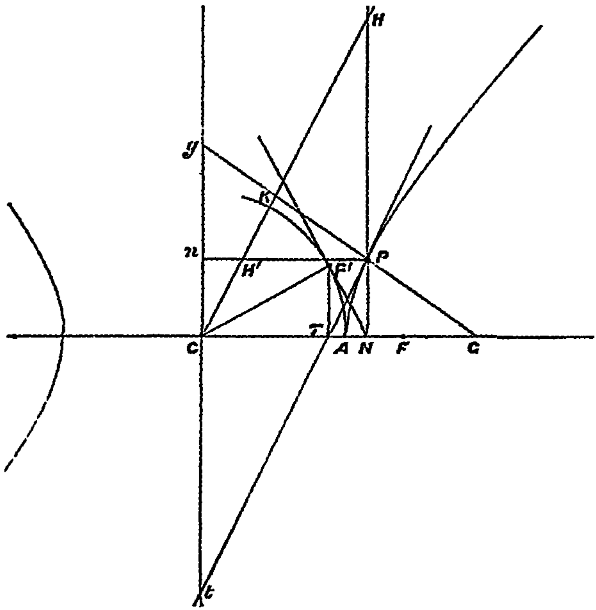
and $\therefore LF : BC :: BC : AC.$

THEOREM 8. SUBTANGENT ON MINOR AXIS

In a central conic, if the tangent at P *meet the conjugate axis in* t, *and* Pn *is perpendicular to that axis,*

$$\mathrm{Cn} \,.\, \mathrm{Ct} = \mathrm{BC}^2$$

Since $$Cn = PN,$$



$$\cdot\ Cn \quad CN :.\ PN \cdot CN,$$

but $$Ct \cdot CT \cdot PN \,.\, TN,$$

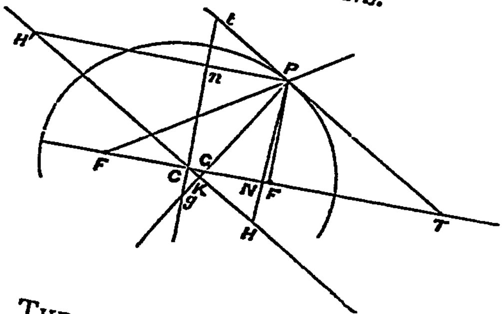
and $\cdot\cdot$ $$Cn \quad Ct \cdot CT \,.\, CN \quad PN^2 \cdot TN \quad CN$$

But $TN \,.\, CN = P'N^2$, where P' corresponds to P,

$$\therefore Cn \quad Ct \,.\, AC^2 \ \cdot PN^2 \cdot P'N^2$$

$$\cdot\ BC^2 \cdot AC^2;$$

$$\therefore Cn \quad Ct = BC^2.$$



THEOREM 9 THE NORMAL.

If in a central conic the normal meets the axes major and minor in G, g, *and* CK *is perpendicular to the normal, then* PG . PK = BC², Pg . PK = AC², *and* CG . CN CF² : AC²

Using the figures of Theorem 8,

Draw PN, Pn perpendicular to the axes, and produce them to meet CK, which is parallel to the tangent at P, in H, H'.

Draw TPt the tangent at P.

Then, since a circle may be described round $KGNH$, the angles at K and N being right angles,

$$PG \,.\, PK = PN \quad PH = Cn \,.\, Ct = BC^2 \qquad \text{(Th 8)}$$

Again, since a circle may be described round $H'nKg$,

$$Pg \quad PK = Pn \,.\, PH' = CN \quad CT = AC^2;$$

$$\therefore \text{ also } PG : Pg \cdot BC^2 : AC^2;$$

but $$GN \quad CN \,.\, GN : Pn,$$

and $$.\!: PG \,.\, Pg,$$

$$\therefore GN \cdot CN \quad BC^2 \cdot AC^2,$$

$$\therefore CG : CN \,.\, AC^2 - BC^2 \cdot AC^2$$

$$.\!: CF^2 : AC^2$$

CHAPTER III.

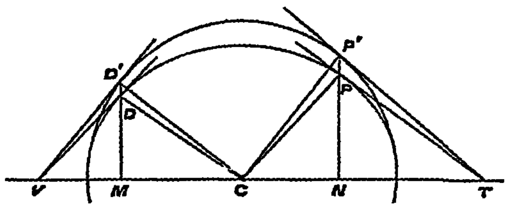
THE ELLIPSE AND HYPERBOLA CONTINUED.

Def A diameter is said to be *conjugate* to another when it is parallel to the tangent at the extremity of the latter

Def An ordinate to a diameter is the line drawn parallel to the tangent at the extremity of that diameter

THEOREM 10 PROPERTIES OF THE ELLIPSE.

In the ellipse if CP *is conjugate to* CD, *then is* CD *conjugate to* CP



Draw the tangents TP, VD at P, D to the ellipse, and at the corresponding points on the auxiliary circle Then CP is given parallel to DV, and it is required to prove CD parallel to PT.

By similar triangles we have

$$VM : MD :: CN : NP,$$

$$\therefore VM : MD' :: CN : NP',$$

therefore VD' is parallel to CP',

and $\therefore P'CD' = CD'V$ is a right angle;

$\therefore P'CD' = TP'C$;

and $\therefore TP'$ is parallel to CD';

$\therefore D'MC$, $P'NT$ are similar,

and $\therefore D'M : MC :: P'N : NT$,

and $\therefore DM : MC :: PN : NT$,

and $\therefore$ the triangles DMC, PNT are similar,

and $\therefore CD$ is parallel to PT.

Cor. 1. The triangles $P'NC$, CMD' are equal in all respects,

and $\therefore CM^2 + CN^2 = P'N^2 + CN^2 = CP'^2 = AC^2$.

Cor. 2. $DM : CN :: BC : AC$.

Cor. 3. $DM^2 + PN^2 = BC^2$,

for $DM^2 : CN^2 :: BC^2 : AC^2$,

and $PN^2 : CM^2 :: BC^2 : AC^2$;

$\therefore DM^2 + PN^2 : CN^2 + CM^2 :: BC^2 : AC^2$,

and $CN^2 + CM^2 = AC^2$;

$\therefore DM^2 + PN^2 = BC^2$.

Cor. 4. $CP^2 + CD^2 = AC^2 + BC^2$.

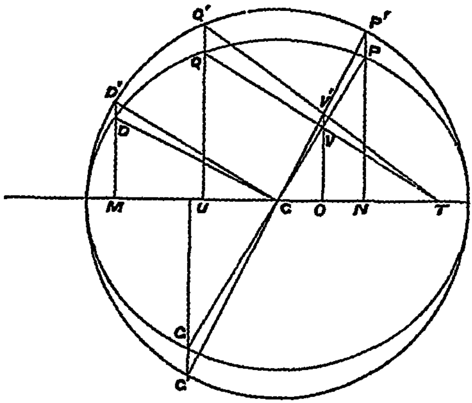
√ THEOREM 11. OBLIQUE ORDINATES AND ABSCISSÆ

In the ellipse if QV *is the ordinate to the diameter* PVG, *and* CD *is conjugate to* CP,

$$QV^2 : PV \cdot VG :: CD^2 : CP^2.$$

Let P', Q', D' be the corresponding points to P, Q, D, join CP', and let the ordinate of V meet CP' in V', so that

$$OV : OV' :: NP : NP' :: UQ : UQ'.$$



Then, by a former Lemma, p 128, QV, $Q'V'$ intersect in the axis at some point T.

And by similar triangles (as in Th. 10) $Q'V'$ may be proved to be parallel to CD', and therefore at right angles to CP';

$$\therefore Q'V'^2 = P'V' \cdot V'G';$$

but $$QV^2 : Q'V'^2 :: CD^2 : CD'^2,$$

and $$PV : P'V' :: CP : CP',$$

and $$VG : V'G' :: CP : CP';$$

$$\therefore PV . VG : P'V' . V'G' :: CP^2 : CP'^2,$$

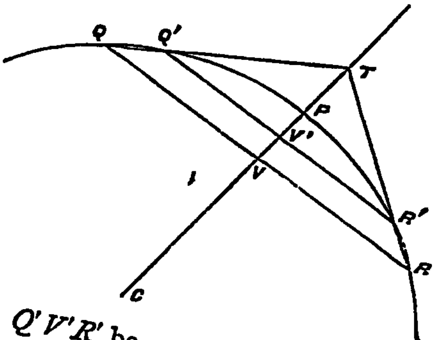
and $\therefore$ since $Q'V'^2 = P'V' . V'G'$ and $CD'^2 = CP'^2$,

$$\therefore QV^2 : PV . VG : CD^2 : CP^2.$$

COR. 1. *Hence all chords of an ellipse are bisected by the diameter to which they are ordinates; and conversely the line that bisects a system of parallel chords is a diameter.*

COR. 2. *If the ordinate and tangent at* Q *meet the diameter in* V, T *respectively,*
then

$$CV . CT = CP^2.$$



Let QVR, $Q'V'R'$ be parallel chords in a central conic, bisected by their conjugate diameter CP in V, V'.

Let QQ' meet CP in T;

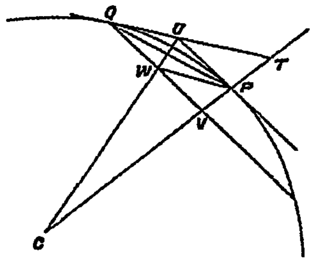
then since $$VT : V'T :: QV : Q'V'$$
$$:: RV : R'V',$$

$\therefore RR'$ also meets CP in T.

Hence when the chords move up to one another, and QT, RT become tangents at Q, R, the tangents at the extremities of a chord intersect on the diameter to which the chord is conjugate

Hence also the line that joins the middle point of a chord with the point of intersection of the tangents at its extremity passes through the centre

Now let QV be an ordinate to CP, QT the tangent at its extremity · then will $CV \cdot CT = CP^2$



Draw PU the tangent at P to QT in U, and PW parallel to QT to meet QV in W.

Then $QUPW$ is a parallelogram, and therefore UW bisects QP, and therefore UW passes through C.

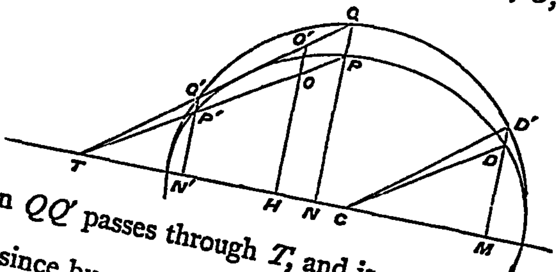
$$\text{Therefore} \quad CV : CP :: CW : CU$$
$$:: CP : CT,$$
$$\text{or } CV \cdot CT = CP^2 \; *$$

* This proof is due to the Rev C Taylor, of St John's College, Cambridge, and is inserted by his permission.

Theorem 12. Rectangles contained by the Segments of Intersecting Chords.

If two chords of an ellipse intersect one another, the rectangles contained by the segments of the chords are proportional to the squares of the diameters parallel to them.

Let POP' be one of the chords through O, CD the parallel semidiameter. Let PP' meet the axis in T, and take Q, O', Q', D' corresponding points to P, O, P', D.



Then QQ' passes through T, and is parallel to CD'.

And since by parallelism

$$PO : QO' :: OP' : O'Q' :: CD : CD',$$

$$\therefore PO \cdot OP' : QO' \cdot O'Q' :: CD^2 : CD'^2,$$

$$\text{or } PO \cdot OP' : CD^2 :: QO' \cdot O'Q' : CD'^2.$$

But if any other chord ROR' were drawn through O, and CS were its parallel semidiameter, then $QO' \cdot O'Q'$, and CD'^2 would remain unaltered, by a property of the circle,

and $$\therefore PO \cdot OP' : CD^2 :: RO \cdot OR' : CS^2,$$

which proves the proposition.

COR. *If a circle intersect an ellipse in the points* P, P′, R, R′, *the chords* PP′, RR′ *must be equally inclined to the axis.*

For if PP', RR' intersect in O, $PO \,.\, OP' = RO \,.\, OR'$ from the circle,

$\therefore CD^2 = CS^2$, or the diameters parallel to the chords are equal, and therefore equally inclined to the axis

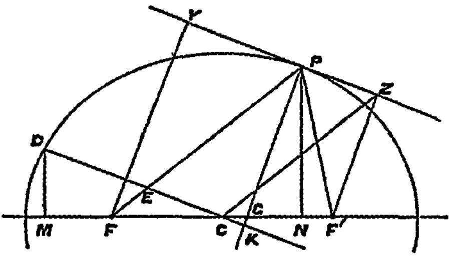
Obs. If one or both of the chords becomes a tangent the rectangle contained by the segments of the chords becomes the square of the tangent.

THEOREM 13

In the ellipse

$$PK \,.\, CD = AC \,.\, BC.$$

Draw DM and PN the ordinates to D, P



Since $\quad PK \,.\, PG = BC^2$, (Theorem 9),

$\therefore PK : BC \,.\, BC \cdot PG.$

But by similar triangles PGN, DMC,

$$PN \quad CM \quad PG : CD$$

But $\quad PN \,.\, CM \,.\, BC \,.\, AC$, (Theorem 10, Cor 2)

$\therefore BC : AC \cdot PG : CD.$

But $\quad PK \,.\, BC \cdot BC \,.\, PG,$

$\therefore$ ex æquali, $PK : AC \cdot BC \quad CD,$

and $\therefore PK \,.\, CD = AC \,.\, BC.$

COR. *Hence the areas of all parallelograms formed by tangents at the extremities of conjugate diameters are equal*

For the area $= 4PK \,.\, CD$

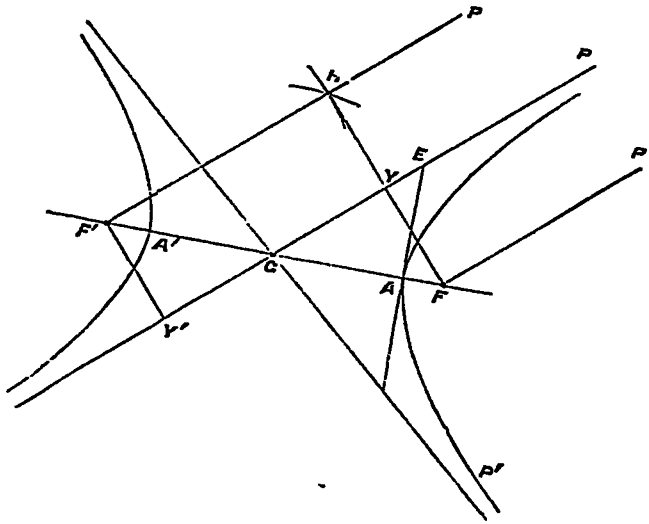
$$= 4AC \,.\, BC.$$

THEOREM 14. PROPERTIES OF THE HYPERBOLA. ASYMPTOTES

Def. A hyperbola whose asymptotes are the same as that of the given hyperbola, and whose conjugate and transverse axes are the transverse and conjugate axes of the latter, is said to be *conjugate* to the latter hyperbola.

Def. A diameter of one hyperbola is said to be *conjugate* to a diameter of the other when it is parallel to the tangent at the extremity of the latter.

Tangents drawn to a hyperbola from its centre meet the curve at an infinite distance from the centre



To draw tangents from C (by the construction given in Theorem 4, Cor 2), describe the director circle with centre F', and a circle with centre C, and radius CF, to intersect the former in H

Then, since $CF = CH = CF'$, FHF' is a right angle.

Draw CY perpendicular to FH, and therefore parallel to $F'H$, and bisecting FH. Then (by Theorem 4), CY touches the curve at the point where CY and $F'H$ intersect

But in this case CY and $F'H$ are parallel, or meet the curve at an infinite distance.

Therefore the tangent from the centre meets the curve at an infinite distance

This tangent is called an *asymptote*, being a line which never meets the curve, though, as will be shewn in the next theorem, it continually approaches to it.

From the symmetry of the curve it is plain that a line equally inclined to the axis on the other side of it is an asymptote to AP', and that these asymptotes produced through the centre are asymptotes to the other branch of the hyperbola.

COR. 1 *The asymptote passes through the intersection of the directrix and the auxiliary circle*

For, since $CY = \frac{1}{2} F'H = CA$, Y is on the auxiliary circle,

And, since YFP is a right angle, Y is on the directrix, FP being drawn to the point of contact (Theorem 3, Cor 3)

COR. 2 *If* AE *is drawn to touch the hyperbola at* A, *and meet the asymptote in* E, AE = BC

For the triangles AEC, YFC are equiangular and have

$$CY = CA, \therefore AE = FY.$$

But $FY \times F'Y' = BC^2$,

and $\therefore FY = BC$,

$\therefore AE = BC$.

It follows that the asymptotes are the diagonals of a rectangle whose sides are the axes, and which touch the vertices of the hyperbola and its conjugate at their middle points.

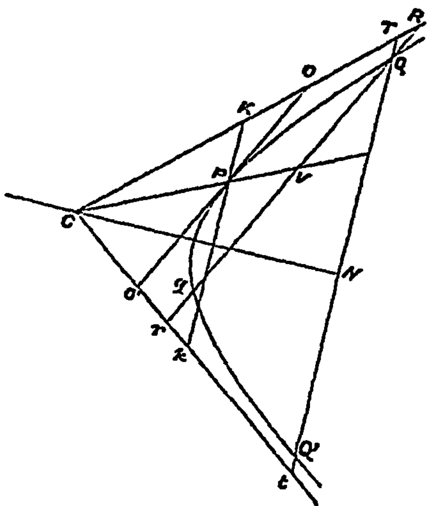
THEOREM 15. SECANTS OF THE CURVE AND ASYMPTOTES.

If TQQ′t, *perpendicular to the axis, cut the asymptotes in* T, t *and the curve in* Q, Q′, *then will*

$$TQ \cdot Qt = BC^2.$$

Let $TQQ't$ meet the axis in N, then $TQ \cdot Qt = TN^2 - QN^2$, and since $TN^2 : CN^2 :: AE^2 : AC^2$ by similar triangles,

$$:: BC^2 : AC^2;$$



and also $QN^2 : AN . A'N . BC^2 \cdot AC^2$, (Th. 7)

$$\therefore TN^2 : QN^2 :: CN^2 : AN \quad A'N$$

$$\cdot \; CN^2 : CN^2 - CA^2$$

$$\therefore TN^2 - QN^2 : QN^2 : AC^2 \cdot CN^2 - CA^2,$$

$$TN^2 - QN^2 : AC^2 \cdot QN^2 . CN^2 - CA^2$$

$$.. \; BC^2 : AC^2, \qquad \text{(Th 7)}$$

$$\therefore TN^2 - QN^2 = BC^2,$$

$$\text{and } \therefore TQ . Qt = BC^2$$

Hence as N moves away from C and Qt becomes greater, TQ becomes less. That is, the line CE perpetually approaches the curve but never meets it, and is therefore called an *asymptote*

THEOREM 16

If OPO′ *be a tangent at* P, *meeting the asymptotes in* O, O′, *and* RQqr *a parallel secant, then will* PO = PO′, RQ = qr, *and* RQ . Qr = PO².

Using the figure of the preceding Theorem, draw $TQQ't$, Kpk perpendicular to the axis.

Then since $RQ : QT . \; PO : PK$,

and $\qquad Qr : Qt :: PO' : Pk$,

$$.. \; RQ . Qr : QT . Qt \; .. \; PO . PO' : PK . Pk,$$

but $\qquad QT . Qt = PK . Pk = BC^2$,

$$\therefore RQ . Qr = PO . PO'.$$

Hence $RQ \cdot Qr = Rq \cdot qr,$

$$\therefore RQ \cdot Qq + RQ \cdot qr = RQ \cdot qr + Qq \cdot qr;$$

$$\therefore RQ = qr;$$

and therefore when the secant becomes a tangent,

$$PO = PO',$$

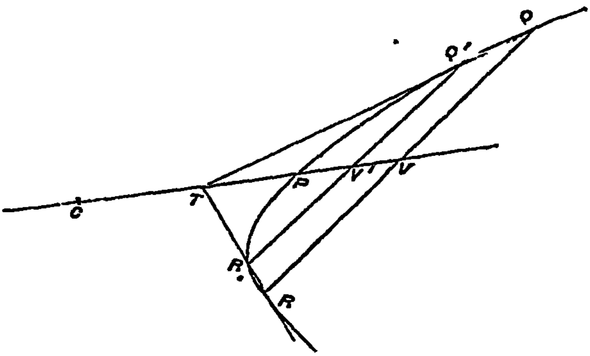
$$\therefore RQ \cdot Qr = PO^2.$$

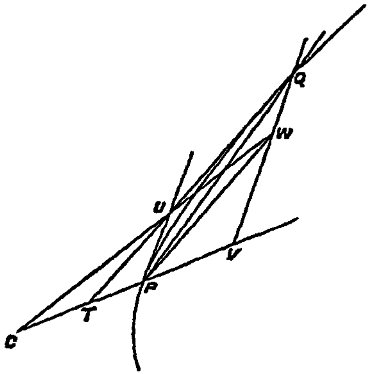
Cor. 1. *The diameter* CP *bisects all the chords parallel to the tangent at* P.

Since $PO = PO'$, $RV = rV$,

and $\therefore QV = qV$.

Cor. 2. Hence $CV \cdot CT = CP^2$ as in the ellipse.





The proof is the same as that given for the ellipse in Th 11, Cor 2.

THEOREM 17 ORDINATE AND ABSCISSA PARALLEL TO ASYMPTOTE.

The rectangle contained by the ordinate and abscissa of any point, measured from the centre parallel to the asymptotes,

$$=\frac{1}{4}(AC^2+BC^2)$$

Draw WPW' perpendicular to the axis

Then, by similar triangles,

$$Pn : PW' \quad CE . 2AE,$$

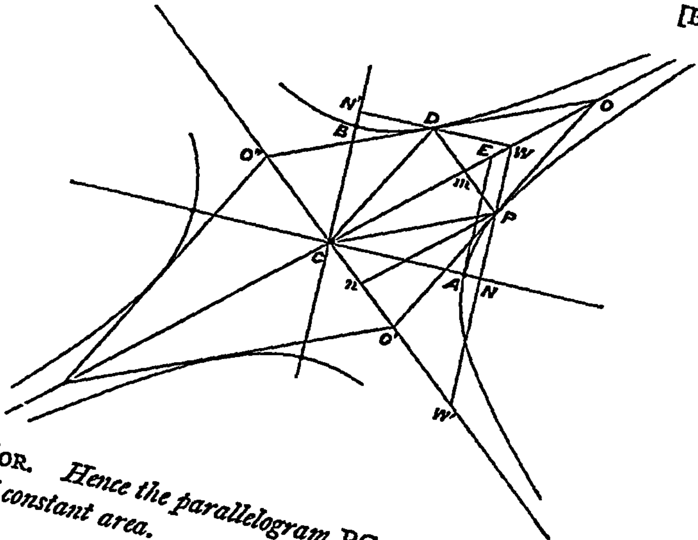
and

$$Pm : PW \cdot CE . 2AE,$$

$$\therefore Pn . Pm : PW . PW' :: CE^2 . 4AE^2,$$

$$\therefore Pn . Pm : BC^2 :: AC^2+BC^2 : 4BC^2,$$

$$\therefore Pn . Pm = \frac{1}{4}(AC^2+BC^2).$$



Cor. *Hence the parallelogram* PC *and the triangle* OCO' *are of constant area.*

Theorem 18. Conjugate Hyperbola.

Tangents at the extremities of conjugate diameters intersect on the asymptotes, and form a parallelogram of constant area = 4AC . BC.

Let OPO', ODO'' be tangents from O a point on the asymptote, meeting the hyperbola and its conjugate in P, D, and making therefore $OP=PO'$ and $OD=DO''$, and therefore PmD parallel to $O'O''$.

Then by the last theorem

$$Pm \cdot mC=\frac{1}{4}(AC^2+BC^2),$$

$$Dm \cdot mC=\frac{1}{4}(BC^2+AC^2);$$

$$\therefore Pm = Dm,$$

and because $$OP = PO',$$

$$\therefore \text{ also } Om = mC,$$

and therefore $ODCP$ is a parallelogram,

$$\therefore CD, CP \text{ are conjugate diameters.}$$

Moreover the area $OO'O''$, which is half the parallelogram formed by the four tangents at the extremities of conjugate diameters, is constant, $= 4PmCn$

But when the tangents are at the vertices the parallelogram becomes a rectangle $= 2AC \times 2BC$;

$$\therefore \text{ the parallelogram} = 4AC \cdot BC$$

COR. 1 *If* PK *is perpendicular to* CD,

$$PK \cdot CD = AC \cdot BC.$$

COR. 2 *Since* PD *is bisected in* m, *and the asymptotes are equally inclined to the axes, therefore parallels to the axes through* P *and* D *intersect on the asymptote.*

For the intercept mW made by both the parallels is equal to mP or mD

COR 3 *Hence also* DN' : PN :: AC . BC,

and CN' . CN :: BC . AC.

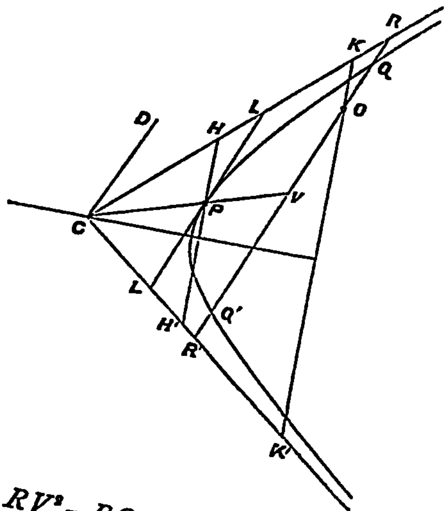
COR 4. $CP^2 - CD^2 = AC^2 - BC^2$.

For determining CP^2 and CD^2 from the triangles CPm, CDm, their difference is proportional to $Cm \cdot mP$, or is constant, and therefore $= AC^2 - BC^2$.

Theorem 19. Oblique Ordinates and Abscissæ.

In the hyperbola $QV^2 : PV \cdot P'V \therefore CD^2 : CP^2$.

Let QV meet the asymptotes in R, R'.



Then $QV^2 = RV^2 - RQ \cdot QR' = RV^2 - CD^2$, (Th 15)

because $PL = CD$.

But $RV : CV :: PL : CP$; (Th. 18)

$\therefore RV^2 : CD^2 :: CV^2 : CP^2$;

$\therefore RV^2 - CD^2 : CD^2 :: CV^2 - CP^2 : CP^2$,

or $QV^2 : CD^2 :: PV \cdot P'V : CP^2$,

and $\therefore QV^2 : PV \cdot P'V :: CD^2 : CP^2$.

THEOREM 20. RECTANGLES CONTAINED BY SEGMENTS OF INTERSECTING CHORDS

If two chords of a hyperbola intersect one another, the rectangles contained by their segments are proportional to the squares of the diameters parallel to them

Let QOQ' be one of the chords through O, in the figure of Theorem 19, meeting the asymptotes in R, R', CD the parallel diameter,

then will $QO \cdot OQ'$ be proportional to CD^2

Draw CPV the diameter to bisect QQ'.

Since $$QO \cdot OQ' = QV^2 - OV^2,$$
and $$RO \cdot OR' = RV^2 - OV^2,$$
$$\therefore\ RO \cdot OR' - QO \cdot OQ' = RV^2 - QV^2$$
$$= RQ \cdot QR'$$
$$= CD^2,$$
$$\therefore\ QO \cdot OQ' = RO \cdot OR' - CD^2.$$

But if KOK' be drawn through O, and HPH' through P, perpendicular to the transverse axis, meeting the asymptotes in K, K', H, H', since

$$RO : KO :: LP : HP,$$
and $$R'O : K'O :: L'P : H'P,$$
$$\therefore\ RO \cdot OR' : KO \cdot OK' :: CD^2 : BC^2,$$

therefore while O remains fixed, and therefore $KO \cdot OK'$ does not alter, $RO \cdot OR'$ is proportional to CD^2,

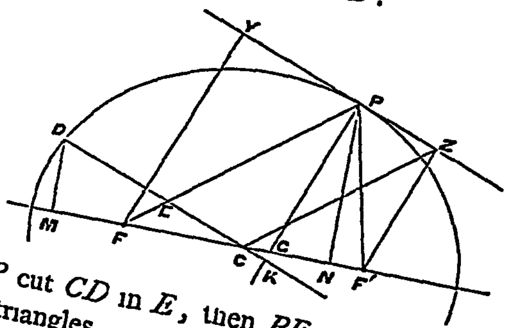
and therefore also $QO \cdot OQ'$ is proportional to CD^2.

Obs. If one or both of the chords becomes a tangent, the rectangle contained by the segments of the chords becomes the square of the tangent

Theorem 21. Product of Focal Distances

In any central conic

$$FP \cdot F'P = CD^2.$$



Let FP cut CD in E, then $PE=CZ=AC$; and since by similar triangles

$$FP : FY :: PE : PK,$$

and

$$F'P : F'Z :: PE : PK;$$

$$\therefore FP \cdot F'P : FY \cdot F'Z :: PE^2 : PK^2,$$

but $FY \cdot F'Z = BC^2$, and $PE=AC$;

$$\therefore FP \cdot F'P : BC^2 :: AC^2 : PK^2,$$

but

$$PK \cdot CD = AC \cdot BC, \text{ (Th. 13 and 18)}$$

$$\therefore CD^2 : BC^2 :: AC^2 : PK^2;$$

$$\therefore FP \cdot F'P : BC^2 :: CD^2 : BC^2;$$

$$\therefore FP \cdot F'P = CD^2.$$

Exercises

1 In the figure of Theorem 1, prove that

$$FF' = AB' \text{ and } AA' = VU$$

2 Shew how to cut from a given cone an ellipse of given axis and eccentricity.

3. Give some mechanical contrivance for describing an ellipse and hyperbola.

4 Prove that

$$CF \cdot CX \quad FC^2 \cdot AC^2$$

in any central conic

5. Prove that in the ellipse $FP + F'P$ is greater or less than AA', according as P is outside or inside the ellipse What is the corresponding property of the hyperbola?

6 If a circle be described on the axis minor of an ellipse as diameter, and $PQ'M$, parallel to the axis major, meet the ellipse in P, the circle in Q' and axis minor in M, prove that

$$Q'M \quad PM \quad BC \,.\, AC$$

7. A circle is described to touch two unequal intersecting circles, prove that the locus of its centre consists of a confocal ellipse and hyperbola.

8 If a hyperbola and ellipse are confocal, they cut one another at right angles.

9 On AB is described a segment of a circle, which is trisected in P, Q Find the locus of P.

10 Prove that the two tangents drawn to a central conic from any point are in the ratio of the parallel diameters

11 Prove that the locus of the point from which tangents can be drawn at right angles to a central conic is a circle whose radius is

$$\sqrt{AC^2 \pm BC^2},$$

the upper sign being taken for the ellipse, and the lower for the hyperbola.

12. Prove that the tangent at the extremity of the latus rectum intersects the axis major at the foot of the directrix, and the axis minor at a point T, such that

$$CT = CA$$

13 Prove that

$$CP + CD > AC + BC,$$

and

$$CP \sim CD < AC - BC.$$

14. Given a central conic to find its centre and axes, foci and directrix.

If an arc of a conic section is given, shew how to find its species.

15 A quadrilateral figure circumscribes an ellipse, prove that its pairs of opposite sides subtend angles at either focus whose sum is two right angles

16. A circle touches an ellipse in P, and cuts it in Q, R, prove that PQ, PR are equally inclined to the axis.

17 If T is the point of intersection of the tangent at P with the tangent at A, prove that FT bisects the angle AFP Hence find the locus of the centres of the escribed circles of the triangle FPF'.

18 If NP produced meet the tangent at the extremity of the latus rectum in T, $TN = FP$.

19 Ellipses are described with a given focus, and to touch a given straight line in a given point, find the locus of the other focus and of the centre

20 Ellipses are described with a given focus, and axis major of given length, to touch a given straight line. find the locus of the other focus, and centre Ans. A circle

21. Ellipses are described with a given focus, and axis minor of given length, to touch a given straight line. to find the locus of the other focus.

Ans A straight line parallel to the given straight line

22. If from the extremities of the axes of an ellipse any four parallel straight lines be drawn, they will intersect the ellipse in the extremities of conjugate diameters.

23 Prove that in an ellipse AP, $A'P$ are parallel to a pair of conjugate diameters, P being any point on the curve

24 A line PFG is constrained to move so that two fixed points in it, F and G, lie on two axes at right angles to one another. Shew that the locus of P is an ellipse.

Hence obtain a mechanical means of drawing an ellipse with given axes

25. An ellipse slides between two lines at right angles to one another; find the locus of its centre. (Th. 5, Cor.)

26. The locus of the points of bisection of chords of an ellipse drawn through a given point is an ellipse of equal eccentricity

27. If the focus of a conic and two points on the curve be given, prove that its directrix will pass through a fixed point. (Th. 3)

28 Given three tangents to an ellipse and one focus, find the other focus (Th 4. Cor. 1.)

29 Prove that the circle FPF' passes through the points of intersection of the tangent and normal at P with the minor axis

30 If CE parallel to the tangent at P meets FP in E, and gE is joined, gE is perpendicular to FP. (Th. 8)

31. With a given focus, and three given points on the curve, find the other focus.

32. The locus of the foot of the perpendicular from the centre on any chord that subtends a right angle at the centre is a circle.

33. Shew that the areas of the ellipse and its auxiliary circle are to one another as $CB : CA$.

34. Chords are drawn to a conic from a fixed point; shew that tangents at their extremities intersect on a fixed straight line.

35. A rifle bullet hits a target. Find the locus of places at which the sound of the discharge and of the hit are heard at the same instant.

36 Given the asymptotes, and one point on the curve, construct for the foci.

37. The corner of a leaf is turned down so that the triangle is of constant area Find the locus of its middle point.

38 Prove by the method of projections that ellipses of equal eccentricity, and whose axes are parallel, can intersect in only two points.

39 A straight line is drawn through a fixed point and is terminated by two given straight lines. find the locus of its middle point

40 If the directrix and focus of an ellipse be fixed, and its axis major continually increased, prove that in the limit the ellipse becomes a parabola. Hence obtain the tangent property of the parabola.

41. The locus of the intersection of tangents to an ellipse at right angles to one another is a circle Deduce the corresponding property in the parabola.

42 The semi-latus rectum is a harmonic mean between the segments of any focal chord

43. If e, e' are the eccentricities of a hyperbola and its conjugate, prove that

$$e\ AC = e'\ BC.$$

44. If F, f are the foci of a hyperbola, and its conjugate, and P, P' conjugate points on the hyperbola and its conjugate,

$$fP' - FP = AC - BC, \text{ and } CF = Cf.$$

45. If any two tangents be drawn to a hyperbola, the lines that join the points where they cut the asymptotes will be parallel.

46. If an ellipse is described with a fixed centre to touch two given straight lines, the locus of its focus is a hyperbola.

47. FY is drawn to make a constant angle FYP with the tangent at P; shew that the locus of Y is a circle.

48. If GK is the perpendicular on SP from G, the foot of the normal at P, PK will be equal to half the latus rectum.

49. A chord of an ellipse which subtends a constant angle at the focus always touches an ellipse with the same focus and directrix.

50. A circle is inscribed in a triangle; prove that if an ellipse be described to touch the three sides of the triangle, and one of its foci is on this circle, the other will be on the same circle.

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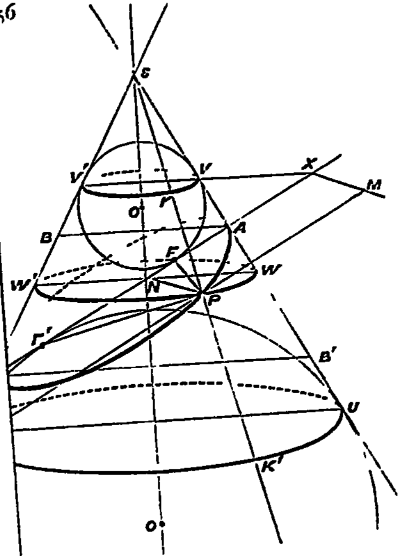
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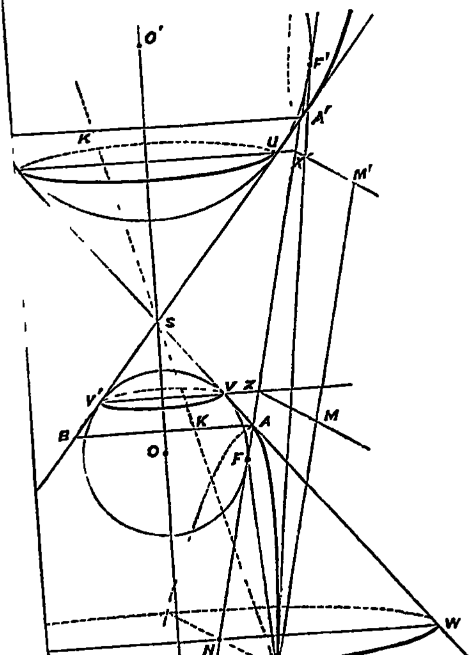
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